LOW-COMPLEXITY ESPRIT METHOD FOR DIRECTION FINDING

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ABSTRACT

A low-complexity ESPRIT method for direction-of-arrival (DOA) estimation is proposed in this paper. Unlike the conventional subspace based methods for DOA estimation, the proposed method only needs the training data of one signal to perform the forward recursions of the multi-stage wiener filter (MSWF), does not involve the estimate of the covariance matrix or its eigendecomposition. Thus, the proposed method is computationally advantageous. Numerical results are given to illustrate the performance of the proposed method.

1. INTRODUCTION

Estimating the direction-of-arrival (DOA) parameters of signals in the noisy background is an important problem in many areas such as communication, radar, sonar and geophysical seismology. It is well known that subspace based methods which rely on the decomposition of the observation space into signal subspace and noise subspace, can provide high-resolution DOA estimations with good performance. However, the common subspace based methods such as the MUSIC [1] and ESPRIT [2] methods, involve the estimate of the covariance matrix and its eigendecomposition. As a result, the common subspace based methods are rather computationally intensive, especially for the case where the model orders in these matrices are large. Recently, the methods called reduced-order correlation kernel estimation technique (ROCKET) [3] and ROCK MUSIC [4] were presented for high-resolution spectrum estimation with lower computational cost. However, the ROCK MUSIC technique needs the forward and backward recursions of the multi-stage wiener filter (MSWF) [5], which increase the computational complexity of the algorithm.

The objective of this paper is to develop a low-complexity method for DOA estimation, based on the MSWF. Unlike the ROCKET or ROCK MUSIC technique, the proposed method merely needs the forward recursions of the MSWF to estimate the signal subspace, does not involve the backward recursions of the MSWF, thereby further reducing the computational complexity of the algorithm. Compared with the more classical eigendecomposition based methods, the proposed method does not need the estimate of the covariance matrix or its eigendecomposition. Therefore, the novel method is computationally advantageous. Basically, the proposed method performs similarly to the classical ESPRIT method but finds the signal subspace in a more computationally efficient way.

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2. PROBLEM FORMULATION

2.1. Data Model

Let us consider a uniform linear array (ULA) consisting of M isotropic sensors. Impinging upon the ULA are P wavefronts from different directions $\{\theta_1, \theta_2, \cdots, \theta_P\}$. The $M \times 1$ output vector of the array at the kth snapshot can be written as

$$\mathbf{x}(k) = \sum_{i=1}^{P} \mathbf{a}(\theta_i) s_i(k) + \mathbf{n}(k) \quad k = 0, \cdots, N-1 \quad (1)$$

where $s_i(k)$ is the scalar complex waveform referred to as the ith signal, $\mathbf{n}(k) \in \mathcal{C}^{M \times 1}$ is the complex noise vector, N denotes the number of snapshots, P represents the number of signals, $\mathbf{a}(\theta_i)$ is the steering vector of the array toward direction θ_i and takes the following form

$$\mathbf{a}(\theta_i) = \frac{1}{\sqrt{M}} \left[1, e^{j\varphi_i}, \cdots, e^{j(M-1)\varphi_i}\right]^T \quad (2)$$

where $\varphi_i = \frac{2\pi d}{\lambda} \sin\theta_i$ in which $\theta_i \in (-\pi/2, \pi/2)$, d and λ are inter-element spacing and the wavelength, respectively.

In matrix notation, (1) can be rewritten as

$$\mathbf{x}(k) = \mathbf{A}(\theta)\mathbf{s}(k) + \mathbf{n}(k) \quad k = 0, 1, \cdots, N-1 \quad (3)$$

where

$$\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_P)] \quad (4)$$

$$\mathbf{s}(k) = [s_1(k), s_2(k), \cdots, s_P(k)]^T \quad (5)$$

are the $M \times P$ steering matrix and the $P \times 1$ complex signal vector, respectively. Throughout the paper we assume that $M > P$. Furthermore, the background noise uncorrelated with the signals is modeled as a stationary, temporally white, zero-mean Gaussian random process, which is

also spatially white and circularly symmetric with the second moments

$$E\left[\mathbf{n}(k)\mathbf{n}^H(l)\right] = \sigma_{\mathbf{n}}^2\delta_{k,l}\mathbf{I}_M \text{ and } E\left[\mathbf{n}(k)\mathbf{n}^T(l)\right] = \mathbf{0} \quad (6)$$

where $\delta_{k,l}$ is the Kronecker delta which is 1 for $k = l$ and 0 for $k \neq l$. We also assume that all signals are jointly stationary, temporally white, zero-mean complex Gaussian random processes. Under these assumptions, the output of the array is complex Gaussian with zero mean and the covariance matrix

$$\mathbf{R}_{\mathbf{x}} = E\left[\mathbf{x}(k)\mathbf{x}^H(k)\right] = \mathbf{A}(\theta)\mathbf{R}_{\mathbf{s}}\mathbf{A}^H(\theta) + \sigma_{\mathbf{n}}^2\mathbf{I}_M \quad (7)$$

where $\sigma_{\mathbf{n}}^2$ is the noise variance and $\mathbf{R}_{\mathbf{s}} = E\left[\mathbf{s}(k)\mathbf{s}^H(k)\right]$ is the signal covariance matrix.

For uncorrelated signals, the eigendecomposition of $\mathbf{R}_{\mathbf{x}}$ can be expressed as

$$\mathbf{R}_{\mathbf{x}} = \mathbf{V}_{\mathbf{s}}\mathbf{\Lambda}_{\mathbf{s}}\mathbf{V}_{\mathbf{s}}^H + \sigma_{\mathbf{n}}^2\mathbf{V}_{\mathbf{n}}\mathbf{V}_{\mathbf{n}}^H = \sum_{i=1}^{M}\lambda_i\mathbf{v}_i\mathbf{v}_i^H \quad (8)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{P+1} = \cdots = \lambda_M = \sigma_{\mathbf{n}}^2$, $\mathbf{V}_{\mathbf{s}} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_P]$ and $\mathbf{V}_{\mathbf{n}} = [\mathbf{v}_{P+1}, \mathbf{v}_{P+2}, \cdots, \mathbf{v}_M]$. Since the column rank of $\mathbf{V}_{\mathbf{s}}$ is equal to the rank P of $\mathbf{R}_{\mathbf{s}}$, the columns of $\mathbf{V}_{\mathbf{s}}$ span the same range subspace of $\mathbf{A}(\theta)$. Considering (7) and (8), and performing some manipulations yields

$$\mathbf{V}_{\mathbf{s}} = \mathbf{A}(\theta)\mathbf{Q} \quad (9)$$

where $\mathbf{Q} \in C^{P\times P}$ is the full-rank matrix.

2.2. Multi-Stage Wiener Filter

The MSWF was developed by Goldstein *et al* [3] to find an approximate solution to the Wiener-Hopf equation which does not need the inverse of the covariance matrix. The MSWF based on the data-level lattice structure [6] is given by the following set of recursions:

- *Initialization*: $d_0(k) = s_1(k)$ and $\mathbf{x}_0(k) = \mathbf{x}(k)$.
- *Forward Recursion*: For $i = 1, 2, \cdots, D$:
  $\mathbf{h}_i = E[\mathbf{x}_{i-1}(k)d_{i-1}^*(k)]/\|E[\mathbf{x}_{i-1}(k)d_{i-1}^*(k)]\|_2$;
  $d_i(k) = \mathbf{h}_i^H\mathbf{x}_{i-1}(k)$;
  $\mathbf{x}_i(k) = \mathbf{x}_{i-1}(k) - \mathbf{h}_i d_i(k)$.
- *Backward Recursion*: For $i = D, D-1, \cdots, 1$ with $\varepsilon_D(k) = d_D(k)$:
  $w_i = E[d_{i-1}(k)\varepsilon_i^*(k)]/E[|\varepsilon_i(k)|^2]$;
  $\varepsilon_{i-1}(k) = d_{i-1}(k) - w_i^*\varepsilon_i(k)$.

3. LOW-COMPLEXITY ESPRIT METHOD

Since the matched filters $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_P$ are mutually orthogonal, the rank P MSWF is completely equivalent to solving the Wiener-Hopf equation in the *Krylov* subspace $\mathcal{K}^{(P)} = span\left\{\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{R}_{\mathbf{x}_0}^{(P-1)}\mathbf{r}_{\mathbf{x}_0 d_0}\right\}$ [7]. Consequently, the matched filters create an orthogonal basis for the *Krylov* subspace. Therefore, there exists a full-rank matrix $\mathbf{K} \in \mathcal{C}^{P\times P}$ such that

$$\mathbf{T}_P = \left[\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{R}_{\mathbf{x}_0}^{(P-1)}\mathbf{r}_{\mathbf{x}_0 d_0}\right]\mathbf{K} \quad (10)$$

where $\mathbf{T}_P = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_P]$. It follows from (8) that

$$\mathbf{R}_{\mathbf{x}_0}^{(i)} = \mathbf{V}_{\mathbf{s}}\mathbf{\Lambda}_{\mathbf{s}}^{(i)}\mathbf{V}_{\mathbf{s}}^H + \sigma_{\mathbf{n}}^{2i}\mathbf{V}_{\mathbf{n}}\mathbf{V}_{\mathbf{n}}^H \quad i = 0, 1, \cdots, P-1 \quad (11)$$

It can be observed that $\mathbf{r}_{\mathbf{x}_0 d_0}$ is contained in the signal subspace. Hence $\mathbf{V}_{\mathbf{n}}^H\mathbf{r}_{\mathbf{x}_0 d_0} = \mathbf{0}$. Inserting (11) into (10) and noting that $\mathbf{V}_{\mathbf{s}}\mathbf{V}_{\mathbf{s}}^H + \mathbf{V}_{\mathbf{n}}\mathbf{V}_{\mathbf{n}}^H = \mathbf{I}_M$ yield

$$\begin{aligned}\mathbf{T}_P &= \left[\mathbf{V}_{\mathbf{s}}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{V}_{\mathbf{s}}\mathbf{\Lambda}_{\mathbf{s}}^{(P-1)}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}\right]\mathbf{K} \\ &= \mathbf{V}_{\mathbf{s}}\left[\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{\Lambda}_{\mathbf{s}}^{(P-1)}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}\right]\mathbf{K}. \quad (12)\end{aligned}$$

By inserting (9) into (12), we have

$$\begin{aligned}\mathbf{T}_P &= \mathbf{A}(\theta)\mathbf{Q}\left[\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{\Lambda}_{\mathbf{s}}^{(P-1)}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}\right]\mathbf{K} \\ &= \mathbf{A}(\theta)\mathbf{H} \quad (13)\end{aligned}$$

where $\mathbf{H} = \mathbf{Q}[\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{\Lambda}_{\mathbf{s}}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{\Lambda}_{\mathbf{s}}^{(P-1)}\mathbf{V}_{\mathbf{s}}^H\mathbf{r}_{\mathbf{x}_0 d_0}]\mathbf{K}$. It can be observed that $\mathbf{H}$ is the nonsingular matrix.

3.1. ESPRIT Method Based on the MSWF

We define the sub-matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ by deleting the first and last rows from $\mathbf{A}(\theta)$ respectively, and the sub-matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ by deleting the first and last rows from $\mathbf{T}_P$ respectively, namely

$$\begin{bmatrix}\mathbf{T}_1 \\ \text{last row}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1 \\ \text{last row}\end{bmatrix}\mathbf{H} = \begin{bmatrix}\mathbf{A}_1\mathbf{H} \\ \text{last row}\end{bmatrix} \quad (14)$$

$$\begin{bmatrix}\text{first row} \\ \mathbf{T}_2\end{bmatrix} = \begin{bmatrix}\text{first row} \\ \mathbf{A}_2\end{bmatrix}\mathbf{H} = \begin{bmatrix}\text{first row} \\ \mathbf{A}_2\mathbf{H}\end{bmatrix}. \quad (15)$$

It follows from (14) and (15) that

$$\mathbf{T}_1 = \mathbf{A}_1\mathbf{H} \quad (16)$$

$$\mathbf{T}_2 = \mathbf{A}_2\mathbf{H}. \quad (17)$$

It is easy to see from (4) that $\mathbf{A}(\theta)$ has the vandermonde structure. Thus, the relation between the sub-matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ can be formulated as

$$\mathbf{A}_2 = \mathbf{A}_1\mathbf{\Phi} \quad (18)$$

where $\mathbf{\Phi} = diag(e^{j\varphi_1}, e^{j\varphi_2}, \cdots, e^{j\varphi_P})$. Therefore, the DOA estimation can be reduced to find the diagonal matrix $\mathbf{\Phi}$. Since $\mathbf{H}$ is the full-rank matrix, by inserting (16)

and (17) into (18), and performing some algebraic manipulations, we have

$$\mathbf{T}_2 = \mathbf{T}_1\mathbf{H}^{-1}\mathbf{\Phi}\mathbf{H} = \mathbf{T}_1\mathbf{\Psi} \tag{19}$$

where $\mathbf{\Psi} = \mathbf{H}^{-1}\mathbf{\Phi}\mathbf{H}$. Notice that $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are related by a similarity transformation, and thus have the same eigenvalues. Solving (19) in either a least-squares sense (LS) or a total-least-squares sense (TLS) yields $\mathbf{\Psi}$. Performing eigendecomposition to $\mathbf{\Psi}$ results in

$$\mathbf{\Psi} = \sum_{i=1}^{P} \mu_i \mathbf{e}_i \mathbf{e}_i^H \tag{20}$$

where μ_i and $\mathbf{e}_i$ are the eigenvalues and eigenvectors of $\mathbf{\Psi}$, respectively. Therefore, the DOA parameters are given by

$$\theta_i = \arcsin\left(\frac{\lambda \arg(\mu_i)}{2\pi d}\right),\ i = 1, 2, \cdots, P \tag{21}$$

where $\arg(\mu_i)$ represents the phase angle of the complex number μ_i.

Remark: Notice that the efficient implementation of MSWF avoids the formation of blocking matrices, and all the operations of the MSWF only involve complex vector-vector products, thereby implying the computational complexity of $O(MN)$for each matched filter $\mathbf{h}_i, i \in \{1, 2, \cdots, P\}$. To fulfil the estimations of the signal subspace, P stages of the MSWF is needed. Thus, the computational cost of the proposed method is only $O(PMN)$ flops. However, the classical ESPRIT method resorts to the estimate of the covariance matrix and its eigendecomposition, which require $O(M^2N + M^3)$ flops. Therefore, the proposed method is much more computationally efficient than the classical ESPRIT method, in particular for the case of large array.

3.2. Extension to Coherent Signal Condition

In the case of coherent signals, we must resort to the smoothing technique to decorrelate the coherent signals since the signal subspace estimated by the method above is incorrect anymore. For the spatial smoothing technique [8], an array of M sensors is subdivided into L subarrays. Thereby, the number of elements per subarray is $M_L = M - L + 1$. For $l = 1, 2, \cdots, L$, let the $M_L \times M$ matrix $\mathbf{J}_l$ be a selection matrix, which takes the following form

$$\mathbf{J}_l = \left[\ \mathbf{0}_{M_L\times(l-1)}\ \vdots\ \mathbf{I}_{M_L\times M_L}\ \vdots\ \mathbf{0}_{M_L\times(M-l-M_L+1)}\ \right]. \tag{22}$$

$\mathbf{J}_l$ is exploited to select part of the observation data matrix $\mathbf{X}_0 = [\mathbf{x}_0(0), \mathbf{x}_0(1), \cdots, \mathbf{x}_0(N-1)]$, which corresponds to the lth subarray. Hence, the spatially smoothed data matrix is constructed as

$$\bar{\mathbf{X}}_0 = [\mathbf{J}_1\mathbf{X}_0\ \ \mathbf{J}_2\mathbf{X}_0\ \ \cdots\ \ \mathbf{J}_L\mathbf{X}_0] \in \mathcal{C}^{M_L\times LN}. \tag{23}$$

Similarly to $\bar{\mathbf{X}}_0$, the "spatially smoothed" training signal vector should take the following form

$$\bar{\mathbf{d}}_0 = [\underbrace{\mathbf{d}_0;\ \mathbf{d}_0;\ \cdots;\ \mathbf{d}_0}_{L}] \in \mathcal{C}^{LN\times 1} \tag{24}$$

where $\mathbf{d}_0 = [d_0(0), d_0(1), \cdots, d_0(N-1)]^T \in \mathcal{C}^{N\times 1}$. Thus, the ith spatially smoothed pre-filter of the MSWF is given by

$$\bar{\mathbf{h}}_i = \frac{\bar{\mathbf{r}}_{\mathbf{x}_{i-1}d_{i-1}}}{\|\bar{\mathbf{r}}_{\mathbf{x}_{i-1}d_{i-1}}\|_2} := \frac{\bar{\mathbf{X}}_{i-1}\mathbf{d}_{i-1}^*}{\|\bar{\mathbf{X}}_{i-1}\mathbf{d}_{i-1}^*\|_2}. \tag{25}$$

Therefore, for coherent signals, the low complexity method for DOA estimation based on the MSWF is summed as follows:

Step1: Perform the spatial smoothing technique to the $M \times N$ observation data matrix $\mathbf{X}_0$, obtain the spatially smoothed $M_L \times NL$ data matrix $\bar{\mathbf{X}}_0$;

Step2: Construct the spatially smoothed training data vector $\bar{\mathbf{d}}_0$ by the way shown in (24);

Step3: Perform the following recursions
For $i = 1, 2, \cdots, P$:
$\bar{\mathbf{h}}_i = \bar{\mathbf{X}}_{i-1}\bar{\mathbf{d}}_{i-1}^*/\|\bar{\mathbf{X}}_{i-1}\bar{\mathbf{d}}_{i-1}^*\|_2$,
$\bar{\mathbf{d}}_i = \bar{\mathbf{h}}_i^H\bar{\mathbf{X}}_{i-1}$,
$\bar{\mathbf{X}}_i = \bar{\mathbf{X}}_{i-1} - \bar{\mathbf{h}}_i\bar{\mathbf{d}}_i$.
Obtain the signal subspace $\bar{\mathbf{T}}_P = \left[\bar{\mathbf{h}}_1, \bar{\mathbf{h}}_2, \cdots, \bar{\mathbf{h}}_P\right]$;

Step4: Define the sub-matrices $\bar{\mathbf{T}}_1$ and $\bar{\mathbf{T}}_2$ in the same way as (14) and (15). Replace $\mathbf{T}_1$ and $\mathbf{T}_2$ of (19) with $\bar{\mathbf{T}}_1$ and $\bar{\mathbf{T}}_2$, respectively. Obtain $\bar{\Psi}$ in either a least-squares sense (LS) or a total-least-squares sense (TLS). Compute the eigenvalues $\bar{\mu}_i$, $i = 1, 2, \cdots, P$ of $\bar{\Psi}$. Obtain the DOA parameters by $\theta_i = \arcsin\left(\frac{\lambda\arg(\bar{\mu}_i)}{2\pi d}\right)$, $i = 1, 2, \cdots, P$.

4. NUMERICAL RESULTS

Suppose there are three signals impinging upon the ULA consisting of 20 sensors from the same signal source. The first is a direct-path signal and the others refer to the scaled and delayed replicas of the first signal that represent the multipaths or the "smart" jammers. The propagation constants are $\{1, -0.8+j0.3, -0.4-j0.5\}$. The true DOAs are also assumed to be $\{0^o, -5^o, 5^o\}$. The rank of the MSWF is equal to 3. 1000 Monte Carlo runs have been made to compute the RMSE's of the estimated DOAs for the two methods.

The RMSE's of the estimated DOAs versus SNR are shown in Fig. 1. It is easy to see from Fig. 1 that the experimental results of the proposed method are nearly identical to those of the classical ESPRIT estimator when SNR$\geq$ 22dB. When SNR is less than 22dB, the proposed method

outperforms its counterpart based on the eigendecomposition of the spatially smoothed covariance matrix. As SNR increases, the RMSE's of the two methods eventually approach to the corresponding Cramér-Rao bound (CRB).

Fig. 2 shows the RMSE's of the estimated DOAs for the two methods versus the number of snapshots. It can be observed that the proposed method surpasses the classical ESPRIT method over the range of the number of snapshots that we simulated. Especially when the number of snapshots is less than 128, the improvement is significant. When the number of snapshots increases, the RMSE's of the two method approach to the CRB.

5. CONCLUSION

We have developed a low-complexity ESPRIT method in this paper. The computational complexity of the proposed method is only $O(PMN)$ flops. In contrast to the classical ESPRIT method which requires $O(M^2N + M^3)$ flops, the proposed method is computationally efficient, especially for the case of large array. Numerical results imply that the proposed method outperforms the classical ESPRIT method in estimation accuracy.

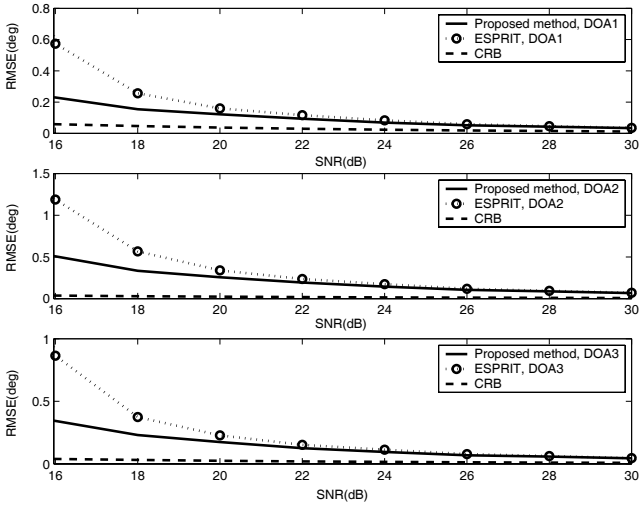

Fig. 1. RMSE's of estimated DOAs versus SNR. N=128, M=20 and 1000 trials.

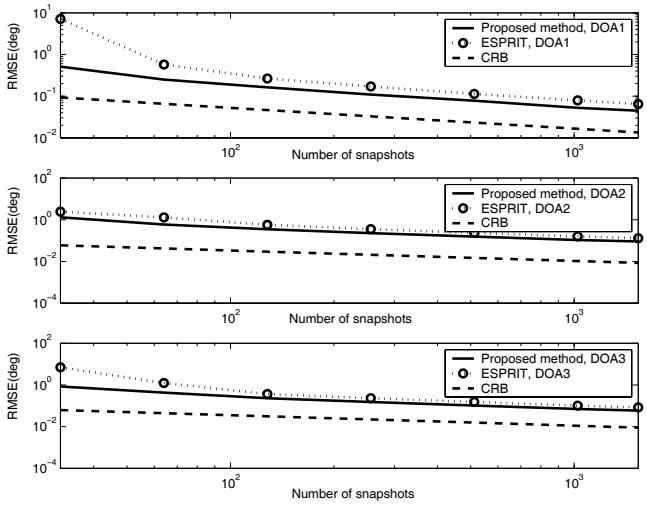

Fig. 2. RMSE's of estimated DOAs versus number of snapshots. M=20, SNR=18dB and 1000 trials.

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