STOCHASTIC CRAMER-RAO BOUND FOR DIRECTION ESTIMATION OF NON-CIRCULAR SIGNALS IN UNKNOWN NOISE FIELDS

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ABSTRACT

This paper gives explicit closed-form expressions of the stochastic Cramer-Rao bound (CRB) on direction of arrival (DOA) estimation accuracy for non-circular Gaussian sources in the case of an arbitrary unknown noise field parameterized by a vector of unknowns from the Slepian-Bangs formula. As a special case, the CRB under the nonuniform white noise assumption is derived. Our expressions can be viewed as extensions of the well-known results by Stoica and Nehorai, Weiss and Friedlander, Ottersten *et al*, and Gershman *et al*. Some properties of these CRBs are proved, then these bounds are numerically compared with the conventional CRBs under the circular complex Gaussian distribution for different unknown noise field models.

1. INTRODUCTION

Stochastic and deterministic CRBs derivation for the DOA parameter alone has been an intensive research field. These bounds have been derived for circular complex Gaussian sources under uniform white noise field in [1, 2, 3, 4] and [5] respectively. Then the stochastic CRB has been derived under nonuniform white and arbitrary unknown parametrized noise fields in[6] and [7] respectively. The general case of an arbitrary unknown noise covariance is particularly important in mobile communications because the dominant noise is external in radio frequency systems [8] and consequently its presence introduces correlation between the different noise processes and because there is normally no signal-free samples available that could be used for estimating the noise covariance. In these applications, non-circular complex signal with discrete distributions are frequently encountered (e.g. binary phase shift keying (BPSK) and offset quadrature phase shift keying (OQPSK) are frequently encountered), but the associated stochastic CRB appears to be prohibitive to compute. Because under rather general conditions, the non-circular complex Gaussian CRB matrix is the largest of all CRB matrices among the class of arbitrary non-circular complex distributions with given covariance matrices (see e.g., [9, p. 293]), we need an explicit expression of the stochastic CRB under non-circular Gaussian distributions of the sources and arbitrary unknown noise fields which can be used as an upper bound of the stochastic CRB under these discrete distributions. Consequently this expression appears to be both an extension of the results [6] and [7] to general non-circular complex Gaussian distributions and result [10] to nonuniform white and arbitrary unknown parametrized noise fields.

In this paper, our derivation is inspired by the proof presented in [6, 7] applied to the extended Slepian-Bangs formula [10]. But, due to the non-circularity of the sources, the key point of this proof, i.e., that the number of terms of the extended source covariance matrix is equal to the number of real and imaginary parts of both sources covariance matrices, is not valid. Consequently to retain the main features of the proof given in [6, 7], we must first prove that the stochastic CRB for the DOA parameter is insensitive to the constraints on this extended covariance matrix. This points will be derived from the study of the ML DOA estimation.

2. ARRAY SIGNAL MODEL

Let an array of M sensors receive K ($K < M$) narrowband signals impinging from the sources with unknown DOAs. The array snapshot complex vectors can be modeled as

$$\mathbf{z}_t = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t + \mathbf{n}_t, \qquad t = 1, \ldots, T$$

where $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}_1, \ldots, \mathbf{a}_K]$ is the full column rank steering matrix where $\mathbf{a}_k$ is parameterized by the scalar θ_k and $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\theta_1, \ldots, \theta_K)^T$. $\mathbf{s}_t = (s_{t,1}, \ldots, s_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by sources and additive noise respectively. $\mathbf{s}_t$ and $\mathbf{n}_t$ are independent, complex zero-mean. $\mathbf{n}_t$ is assumed circular complex Gaussian with unknown covariance matrix $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \mathbf{Q}_n$, while $\mathbf{s}_t$ is non-circular complex Gaussian, and possibly spatially correlated or even coherent with $\mathbf{R}_s \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_t\mathbf{s}_t^H)$ and $\mathbf{R}'_s \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_t\mathbf{s}_t^T)$. This leads to the covariance matrices of $\mathbf{z}_t$:

$$\mathbf{R}_z(\boldsymbol{\alpha}) = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{Q}_n \quad \text{and} \quad \mathbf{R}'_z(\boldsymbol{\alpha}) = \mathbf{A}\mathbf{R}'_s\mathbf{A}^T,$$

where the vector $\boldsymbol{\alpha}$ of unknown real parameters collects the DOAs and nuisance parameters. These covariance matrices are estimated by $\mathbf{R}_{z,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{z}_t\mathbf{z}_t^H$ and $\mathbf{R}'_{z,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{z}_t\mathbf{z}_t^T$, respectively. Let us consider the following general noise model introduced in [11] and used in [7]

$$\mathbf{Q}_n = \mathbf{Q}_n(\boldsymbol{\sigma})$$

where $\boldsymbol{\sigma} \stackrel{\text{def}}{=} (\sigma_1 \ldots, \sigma_N)^T$ is the vector of real unknown coefficients which are used to parameterize the noise covariance matrix. If no a priori information is available concerning the spatial covariances of the sources, $(\mathbf{R}_s, \mathbf{R}'_s)$ is generically parameterized by the real parameters $\boldsymbol{\rho} = ((\Re([\mathbf{R}_s]_{i,j}), \Im([\mathbf{R}_s]_{i,j}), \Re([\mathbf{R}'_s]_{i,j}), \Im([\mathbf{R}'_s]_{i,j}))_{1\leq j<i\leq K}, ([\mathbf{R}_s]_{i,i}, \Re([\mathbf{R}'_s]_{i,i}), \Im([\mathbf{R}'_s]_{i,i}))_{i=1,\ldots,K})^T$. Thus the vector of unknown real parameters can be written as $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\boldsymbol{\theta}^T, \boldsymbol{\rho}^T, \boldsymbol{\sigma}^T)^T$. This parameter is supposed identifiable from $(\mathbf{R}_z(\boldsymbol{\alpha}), \mathbf{R}'_z(\boldsymbol{\alpha}))$. The PDF of $\mathbf{z}_t$ can be expressed as a function of $\tilde{\mathbf{z}}_t \stackrel{\text{def}}{=} (\mathbf{z}_t^T, \mathbf{z}_t^{*T})^T$ as

$$p(\tilde{\mathbf{z}}_t) = (\pi)^{-M}[\text{Det}(\mathbf{R}_{\tilde{z}}(\boldsymbol{\alpha}))]^{-1/2}\exp[-\frac{1}{2}\tilde{\mathbf{z}}_t^H\mathbf{R}_{\tilde{z}}^{-1}(\boldsymbol{\alpha})\tilde{\mathbf{z}}_t] \tag{2.1}$$

where $\mathbf{R}_{\tilde{z}}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \text{E}(\tilde{\mathbf{z}}_t\tilde{\mathbf{z}}_t^H) = \tilde{\mathbf{A}}(\boldsymbol{\theta})\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H(\boldsymbol{\theta}) + \mathbf{Q}_{\tilde{n}}$ with $\mathbf{R}_{\tilde{s}} = \begin{bmatrix} \mathbf{R}_s & \mathbf{R}'_s \\ \mathbf{R}'^*_s & \mathbf{R}^*_s \end{bmatrix}$, $\tilde{\mathbf{A}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A}(\boldsymbol{\theta}) & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^*(\boldsymbol{\theta}) \end{bmatrix}$ and $\mathbf{Q}_{\tilde{n}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{Q}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}^*_n \end{bmatrix}$.

3. STOCHASTIC CRAMER-RAO BOUNDS

To derive the stochastic CRB of the parameter $\boldsymbol{\theta}$ alone, two approaches could be considered. One of them consists in computing the asymptotic covariance matrix of the ML estimator, and the other is obtained directly from the extended Slepian-Bangs formula derived in [10]. The first approach has been successfully used in the case of uniform white noise fields in [1, 10], where a closed-form expression of the log-likelihood function concentrated with respect to the full set of the signal and noise nuisance parameters was available. In the case of nonuniform white and linearly parameterized noise fields, such property has appeared to be impossible to obtain in [6] and [12] respectively. Consequently, we concentrate on the second approach. To adapt the proofs given in the circular Gaussian case in [2, 6] and [7] in the uniform white, nonuniform white and arbitrary unknown parameterized noise field respectively, to the non-circular case, the key point $\text{Vec}(\mathbf{R}_{\tilde{s}}) = \mathbf{J}\boldsymbol{\rho}$ where $\mathbf{J}$ is a constant nonsingular matrix must be preserved. Because $\mathbf{R}_{\tilde{s}}$ is structured, we must first prove that the stochastic CRB for the DOA parameter is insensitive to the constraints on $\mathbf{R}_{\tilde{s}}$.

3.1. Maximum likelihood estimation

We first note that the log-likelihood function associated with the PDF (2.1) can be classically written (see e.g. [1]) after dropping the constants as

$$L(\boldsymbol{\theta}, \boldsymbol{\rho}, \boldsymbol{\sigma}) = -\frac{T}{2}\left(\ln[\text{Det}(\mathbf{R}_{\tilde{z}})] + \text{Tr}(\mathbf{R}_{\tilde{z}}^{-1}\mathbf{R}_{\tilde{z},T})\right) \tag{3.1}$$

Due to the structures of $\mathbf{R}_{\tilde{s}}$ and $\mathbf{Q}_{\tilde{n}}$ in $\mathbf{R}_{\tilde{z}}$, the ML estimation of $(\boldsymbol{\theta}, \boldsymbol{\rho}, \boldsymbol{\sigma})$ becomes a constrained optimization problem which is not standard. Despite this difficulty, we prove in [13] the following result

Result 1 *If the sample covariance matrix $\mathbf{R}_{\tilde{z},T}$ is positive definite, the joint constrained and unconstrained ML estimates which maximize the log-likelihood function* (3.1) *coincide.*

3.2. Stochastic Cramer-Rao bound expressions

From the previous result, the stochastic CRB for the signal DOAs associated with the constrained and unconstrained array signal models coincide. Using the unconstrained model, let $\boldsymbol{\alpha} = (\boldsymbol{\theta}^T, \boldsymbol{\rho}^T, \boldsymbol{\sigma}^T)^T$ with here $\boldsymbol{\rho}$ contains the real parameters of the unconstrained matrix $\mathbf{R}_{\tilde{s}}$. With this unconstrained model, we can follow along the lines of the derivation given in [7] where $\mathbf{R}_z = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{Q}_n$ is replaced here by $\mathbf{R}_{\tilde{z}} = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H + \mathbf{Q}_{\tilde{n}}$ because the key point of the derivation, i.e., the relation $\text{Vec}(\mathbf{R}_{\tilde{s}}) = \mathbf{J}\boldsymbol{\rho}$ where $\mathbf{J}$ is a constant nonsingular complex matrix is preserved. By adapting the proof given in [7], the following result is proved in [13].

Result 2 *The normalized (i.e., for $T = 1$) DOA-related block of CRB for non-circular complex Gaussian (NCG) sources in the presence of an arbitrary unknown (AU) noise field is given by the following explicit expression:*

$$\mathbf{CRB}_{\text{AU}}^{\text{NCG}}(\boldsymbol{\theta}) = \frac{1}{2}\left\{\Re\left[\left(\breve{\mathbf{D}}^H\boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}\breve{\mathbf{D}}\right) \odot \left([\mathbf{R}_s\breve{\mathbf{A}}^H, \mathbf{R}'_s\breve{\mathbf{A}}^T]\,\bar{\mathbf{R}}_{\tilde{z}}^{-1}\begin{bmatrix} \breve{\mathbf{A}}\mathbf{R}_s \\ \breve{\mathbf{A}}^*\mathbf{R}'^*_s \end{bmatrix}\right)^T\right] - \mathbf{M}_{\text{AU}}^{\text{NCG}}\mathbf{T}_{\text{AU}}^{\text{NCG}\,-1}\mathbf{M}_{\text{AU}}^{\text{NCG}\,T}\right\}^{-1} \tag{3.2}$$

using real matrices

$$\begin{aligned}
\mathbf{M}_{\text{AU}}^{\text{NCG}} &= 2\Re\left\{\mathcal{Q}^T\left[(\breve{\mathbf{D}}^H\boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}) \otimes (\mathbf{G}\breve{\mathbf{A}}\mathbf{R}_s)^T\right]\mathcal{P}^*\right\} \\
&\quad +2\Re\left\{\mathcal{Q}^T\left[(\breve{\mathbf{D}}^H\boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}) \otimes (\mathbf{G}'\breve{\mathbf{A}}^*\mathbf{R}'^*_s)^T\right]\mathcal{P}^*\right\} \\
\mathbf{T}_{\text{AU}}^{\text{NCG}} &= 4\Re\left\{\mathcal{P}^H\left[\mathbf{G}^T \otimes \boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}\right]\mathcal{P}\right\} \\
&\quad -2\left(\mathcal{P}^H\left[(\boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp})^T \otimes \boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}\right]\mathcal{P}\right)
\end{aligned}$$

with $\mathcal{Q} \stackrel{\text{def}}{=} \left[\text{vec}(\mathbf{e}_1\mathbf{e}_1^T), \text{vec}(\mathbf{e}_2\mathbf{e}_2^T), \ldots, \text{vec}(\mathbf{e}_K\mathbf{e}_K^T)\right]$ *and* $\mathcal{P} \stackrel{\text{def}}{=} \left[\text{vec}(\bar{\mathbf{Q}}_n^1), \text{vec}(\bar{\mathbf{Q}}_n^2), \ldots, \text{vec}(\bar{\mathbf{Q}}_n^N)\right]$ *where* $\mathbf{e}_i$ *contains one in the ith position and zeros elsewhere and* $\mathbf{Q}_n^k \stackrel{\text{def}}{=} \frac{d\mathbf{Q}_n(\boldsymbol{\sigma})}{d\sigma_k}$, $\bar{\mathbf{Q}}_n^k \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}\mathbf{Q}_n^k\mathbf{Q}_n^{-1/2}$, $\breve{\mathbf{A}} \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}\mathbf{A}$, $\breve{\mathbf{D}} \stackrel{\text{def}}{=} \frac{d\breve{\mathbf{A}}}{d\boldsymbol{\theta}}$, $\mathbf{D} \stackrel{\text{def}}{=} [\mathbf{d}_1, \ldots, \mathbf{d}_K] \stackrel{\text{def}}{=} \left[\frac{d\mathbf{a}_1}{d\theta_1}, \ldots, \frac{d\mathbf{a}_K}{d\theta_K}\right]$, $\bar{\mathbf{R}}_{\tilde{z}} \stackrel{\text{def}}{=} \mathbf{Q}_{\tilde{n}}^{-1/2}\mathbf{R}_{\tilde{z}}\mathbf{Q}_{\tilde{n}}^{-1/2}$, $\bar{\mathbf{R}}_z \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}\mathbf{R}_z\mathbf{Q}_n^{-1/2}$, $\bar{\mathbf{R}}'_z \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}\mathbf{R}'_z\mathbf{Q}_n^{-1/2}$, $\mathbf{G} = \left(\bar{\mathbf{R}}_z - \bar{\mathbf{R}}'_z\bar{\mathbf{R}}_z^{*-1}\bar{\mathbf{R}}'^*_z\right)^{-1}$ *and* $\mathbf{G}' = -\mathbf{G}\bar{\mathbf{R}}'_z\bar{\mathbf{R}}_z^{*-1}$.

When the noise is spatially uncorrelated with different sensor noise variances (nonuniform white noise (NU)). Result 2 takes the following form that is proved in [13].

Result 3 *For non-circular complex Gaussian sources, the normalized DOA-related block of CRB under the nonuniform white noise assumption is given by:*

$$\mathbf{CRB}_{\text{NU}}^{\text{NCG}}(\boldsymbol{\theta}) = \frac{1}{2}\left\{\Re\left[\left(\breve{\mathbf{D}}^H\boldsymbol{\Pi}_{\breve{\mathbf{A}}}^{\perp}\breve{\mathbf{D}}\right) \odot \left([\mathbf{R}_s\breve{\mathbf{A}}^H, \mathbf{R}'_s\breve{\mathbf{A}}^T]\,\bar{\mathbf{R}}_{\tilde{z}}^{-1}\begin{bmatrix} \breve{\mathbf{A}}\mathbf{R}_s \\ \breve{\mathbf{A}}^*\mathbf{R}'^*_s \end{bmatrix}\right)^T\right] - \mathbf{M}_{\text{NU}}^{\text{NCG}}\mathbf{T}_{\text{NU}}^{\text{NCG}\,-1}\mathbf{M}_{\text{NU}}^{\text{NCG}\,T}\right\}^{-1} \tag{3.3}$$

using real matrices

$$\mathbf{M}_{\mathrm{NU}}^{\mathrm{NCG}} = 2\Re\left[(\breve{\mathbf{D}}^H\mathbf{\Pi}_{\breve{\mathbf{A}}}^{\perp}) \odot (\mathbf{G}\breve{\mathbf{A}}\mathbf{R}_s)^T + (\breve{\mathbf{D}}^H\mathbf{\Pi}_{\breve{\mathbf{A}}}^{\perp}) \odot (\mathbf{G}'\breve{\mathbf{A}}^*\mathbf{R}_s'^*)^T\right]$$

$$\mathbf{T}_{\mathrm{NU}}^{\mathrm{NCG}} = 2\left(\mathbf{G}^T \odot \mathbf{G} - (\mathbf{\Pi}_{\breve{\mathbf{A}}}\mathbf{G})^T \odot (\mathbf{\Pi}_{\breve{\mathbf{A}}}\mathbf{G})\right).$$

3.3. Single source case

In the particular case of one signal source, it is shown in [13] that the CRB given by Result 3 can be simplified to

Result 4 *The CRB of θ_1 for a non-circular complex Gaussian source corrupted by nonuniform white noise field decreases monotonically as the non-circularity rate increases and is given by*

$$CRB_{\mathrm{NU}}^{\mathrm{NCG}}(\theta_1) = \frac{1}{\alpha_1}\left[\frac{2r_1^{-1} + \|\mathbf{a}_1\|^{-2}r_1^{-2} + \|\mathbf{a}_1\|^2 - \|\mathbf{a}_1\|^2\rho_1^2}{\|\mathbf{a}_1\|^2 r_1 + 1 + (1 - \|\mathbf{a}_1\|^2 r_1)\rho_1^2}\right]$$

where the non-circularity rate ρ_1 is defined by $\mathrm{E}(s_{t,1}^2) = \rho_1 e^{i\phi_1}\mathrm{E}|s_{t,1}^2|$ and satisfies $0 \le \rho_1 \le 1$ (ϕ_1 is the circularity phase of $s_{t,1}$). The SNR is defined as in [6, rel. (48)] by $r_1 \stackrel{\mathrm{def}}{=} \frac{\sigma_{s_1}^2}{M}\sum_{i=1}^M \frac{1}{\sigma_i^2}$ where $\sigma_{s_1}^2 \stackrel{\mathrm{def}}{=} \mathrm{E}|s_{t,1}^2|$, and α_1 is the noise dependant factor $\left(\frac{2\|\mathbf{a}_1\|^2}{\mathbf{a}_1^H\mathbf{Q}_n^{-1}\mathbf{a}_1}\right)\breve{\mathbf{d}}_1^H\mathbf{\Pi}_{\breve{\mathbf{a}}_1}^{\perp}\breve{\mathbf{d}}_1$ with $\breve{\mathbf{a}}_1 = \mathbf{Q}_n^{-1/2}\mathbf{a}_1$ and $\breve{\mathbf{d}}_1 = \frac{d\breve{\mathbf{a}}_1}{d\theta_1}$. For $\rho_1 = 0$, we note that an expression of $CRB_{\mathrm{NU}}^{\mathrm{CG}}(\theta_1)$ has already been given [6, rel. (46)], but with a more intricate expression.

4. COMPARISONS BETWEEN CRBS

Let us consider the situation when the noise is uniform and spatially white (U) while yet, not knowing this, the noise is modeled using $N > 1$ parameters. Comparing (3.2) and (3.3) under these conditions with

$$\mathbf{CRB}_{\mathrm{U}}^{\mathrm{NCG}}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2}\left\{\Re\left[\left(\mathbf{D}^H\mathbf{\Pi}_{\mathbf{A}}^{\perp}\mathbf{D}\right) \odot \left([\mathbf{R}_s\mathbf{A}^H, \mathbf{R}_s'\mathbf{A}^T]\mathbf{R}_{\tilde{z}}^{-1}\begin{bmatrix}\mathbf{A}\mathbf{R}_s \\ \mathbf{A}^*\mathbf{R}_s'^*\end{bmatrix}\right)^T\right]\right\}^{-1}$$

obtained in [10], we have because $\mathbf{M}_{(\times)}^{\mathrm{NCG}}\mathbf{T}_{(\times)}^{\mathrm{NCG}\,-1}\mathbf{M}_{(\times)}^{\mathrm{NCG}\,T}$ is nonnegative definite

$$\mathbf{CRB}_{\mathrm{AU}}^{\mathrm{NCG}}(\boldsymbol{\theta})|_{\mathbf{Q}_n=\sigma_n^2\mathbf{I}_M} \ge \mathbf{CRB}_{\mathrm{U}}^{\mathrm{NCG}}(\boldsymbol{\theta})$$

and $$\mathbf{CRB}_{\mathrm{NU}}^{\mathrm{NCG}}(\boldsymbol{\theta})|_{\mathbf{Q}_n=\sigma_n^2\mathbf{I}_M} \ge \mathbf{CRB}_{\mathrm{U}}^{\mathrm{NCG}}(\boldsymbol{\theta}).$$

Let us now compare the stochastic and asymptotic deterministic CRBs in the case of colored or nonuniform white noise field for non-circular complex source signals. First, we note that the following expression of the asymptotic deterministic CRB proved in [6] in the circular case remains valid in the non-circular case as well

$$\mathbf{CRB}_{\mathrm{AU}}^{\mathrm{DET}}(\boldsymbol{\theta}) = \frac{1}{2}\left\{\Re\left[(\breve{\mathbf{D}}^H\mathbf{\Pi}_{\breve{\mathbf{A}}}^{\perp}\breve{\mathbf{D}}) \odot \mathbf{R}_s^T\right]\right\}^{-1}.$$

We prove in [13], the following result

Result 5 *If $\mathbf{R}_{\tilde{s}}$ is nonsingular*

$$\mathbf{CRB}_{\mathrm{AU}}^{\mathrm{DET}}(\boldsymbol{\theta}) \le \mathbf{CRB}_{\mathrm{AU}}^{\mathrm{NCG}}(\boldsymbol{\theta}).$$

5. ILLUSTRATIVE EXAMPLES

The purpose of this section is to illustrate Results 2, 3 and 5, and to compare these stochastic CRBs to the stochastic CRBs under circular complex Gaussian distributed source signals as well with the deterministic CRB. We consider throughout this section two independent and equipowered sources with identical non-circularity rate. These sources impinge on a uniform linear array of $M = 10$ sensors for which $\mathbf{a}_k = (1, e^{i\theta_k}, \ldots, e^{i(M-1)\theta_k})^T$. We assume that the noise field is modeled by the three following covariance matrices $\mathbf{Q}_n^{(i)}, i = 1, 2, 3$. The first two models and the third model come from [7] and [6] respectively.

$$\mathbf{Q}_n^{(1)}(k,l) = \sigma_n^2\exp(-(k-l)^2\zeta)$$
$$\mathbf{Q}_n^{(2)}(k,l) = \sigma_n^2\exp(-|k-l|\zeta)$$
$$\mathbf{Q}_n^{(3)} = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_M^2).$$

In the first two colored noise field models, $\boldsymbol{\sigma} = (\sigma_n^2, \zeta)^T$ where ζ is the 'color' parameter and the SNR is defined by $\frac{\sigma_{s_1}^2}{\sigma_n^2}$ and in the nonuniform white noise field model $\boldsymbol{\sigma} = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_{10}^2)^T$ and the SNR is defined by $\frac{\sigma_{s_1}^2}{10}\sum_{i=1}^{10}\frac{1}{\sigma_i^2}$.

In Fig. 1, we compare the stochastic CRBs under circular and non-circular complex Gaussian distributed source signals to the deterministic CRB. The first two noise field models are used. The bounds $CRB_{\mathrm{AU}}^{\mathrm{NCG}}(\theta_1)$, $CRB_{\mathrm{AU}}^{\mathrm{CG}}(\theta_1)$ and $CRB_{\mathrm{AU}}^{\mathrm{DET}}(\theta_1)$ [1] are plotted against ζ. Compared to [7, Fig. 1], we note a similarity of behavior of these CRBs. We note that when ζ decreases all the CRBs approach zero because $\mathbf{Q}_n$ becomes singular. When $\zeta \gg 1$, the two first noise model transform to the uniform white noise model and each of the three CRBs associated with the two models merges. We see that the stochastic CRB under non-circular complex Gaussian distributed sources is visibly larger that the deterministic CRB.

In Figs. 2 and 3, we compare the non-circular Gaussian CRB with the circular Gaussian CRB by means of the ratio $r \stackrel{\mathrm{def}}{=} \frac{CRB_{\mathrm{NU}}^{\mathrm{NCG}}(\theta_1)}{CRB_{\mathrm{NU}}^{\mathrm{CG}}(\theta_1)}$ for the third noise model. These figures examine the dependence of the ratio r with the non-circularity rate $\rho_2 = \rho_1$, the DOA separation $\Delta\theta = \theta_2 - \theta_1$ and the SNR. Fig. 2 shows that $CRB_{\mathrm{NU}}^{\mathrm{NCG}}(\theta_1)$ decreases as the non-circularity rate increases (this extends to two equipowered and independent sources a property proved in the one source case). Furthermore this decrease is more prominent for low DOA separation. Fig. 3 shows that r decreases as the DOA separation and the SNR decrease and the difference of order of magnitude between $CRB_{\mathrm{NU}}^{\mathrm{NCG}}(\theta_1)$ and $CRB_{\mathrm{NU}}^{\mathrm{CG}}(\theta_1)$ is quite significant for low DOA separations and SNRs.

[1] All the CRBs are computed for $T = 1$.

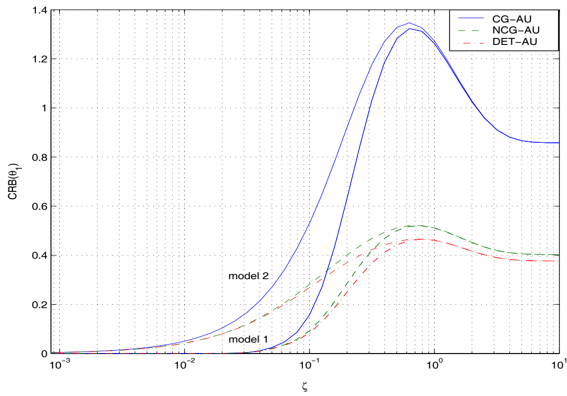

Fig.1 $CRB^{\mathrm{NCG}}_{\mathrm{AU}}(\theta_1)$, $CRB^{\mathrm{CG}}_{\mathrm{AU}}(\theta_1)$ and $CRB^{\mathrm{DET}}_{\mathrm{AU}}(\theta_1)$ as a function of ζ with $\Delta\theta = 0.1rd$, SNR= $0dB$ and$\Delta\phi = 0.52rd$

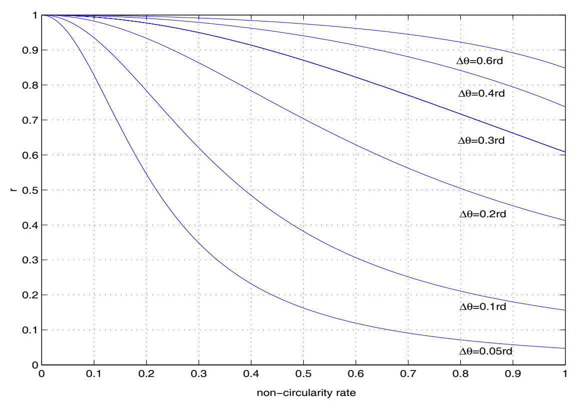

Fig.2 Ratio $r \stackrel{\mathrm{def}}{=} \frac{CRB^{\mathrm{NCG}}_{\mathrm{NU}}(\theta_1)}{CRB^{\mathrm{CG}}_{\mathrm{NU}}(\theta_1)}$ with SNR= $-5dB$ and $\Delta\phi = 0.52rd$

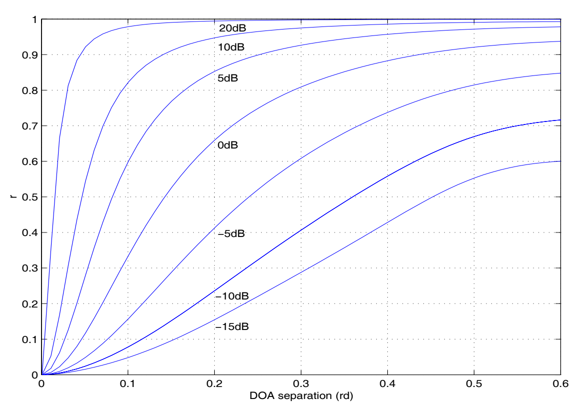

Fig.3 Ratio $r \stackrel{\mathrm{def}}{=} \frac{CRB^{\mathrm{NCG}}_{\mathrm{NU}}(\theta_1)}{CRB^{\mathrm{CG}}_{\mathrm{NU}}(\theta_1)}$ with $\rho_2 = \rho_1 = 1$ and $\Delta\phi = 0.52rd$

6. CONCLUSION

New closed-form expressions of the stochastic CRBs of the DOA parameter estimates for non-circular complex Gaussian sources in the general case of an arbitrary unknown noise field have been presented. Compared with the deterministic CRB and the circular complex Gaussian CRB, some properties have been proved and some numerical examples with particular noise fields have been exhibited. They show that the difference between the non-circular and circular complex Gaussian CRB may be quite significant, particularly for low DOA separations and SNRs. Consequently our derived non-circular complex Gaussian CRB provides a tighter upper bound on the CRB under non-circular complex discrete distribution compared to the standard circular complex Gaussian CRB.

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