SINGULAR VALUE DECOMPOSITION OF A MATRIX-VALUED IMPULSE RESPONSE

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ABSTRACT

The singular value decomposition (svd), a standard tool for theoretical and computational matrix analysis, represents a matrix as an ordered product of a unitary matrix, a nonnegative, real diagonal matrix, and a second unitary matrix. Our contribution is to extend the svd representation to a matrix-valued impulse response. The representation involves a countably infinite number of vector-valued eigenfunctions and scalar singular values, and it provides the most concise and elegant description of the action of a matrix-valued filter (for example, a space-time communication channel) when driven by a finite-duration input signal.

1. INTRODUCTION

The singular value decomposition (svd) represents a $N \times M$ complex matrix H as a product, $H = \Phi D_v \Psi^\dagger$, where D_v is a diagonal square matrix having row and column dimensions $\min(M, N)$ whose diagonal elements $\{v_\ell\}$, the *singular values*, are real and nonnegative, and Φ and Ψ are unitary matrices having row-dimensions of N and M respectively, and column dimensions equal to $\min(M, N)$ [6]. The superscript "†" denotes "conjugate transpose". The svd provides the simplest way to visualize the action of matrix multiplication, and it is a natural theoretical and computational tool for formulating problems of over- and under-determined systems of linear equations.

Our contribution is to extend the svd representation to the *matrix-valued impulse response* of a space-time linear channel. The mathematical basis for the svd is the representation by Kelly and Root [1] of a nonnegative-definite matrix-valued kernel in terms of scalar eigenvalues and vector eigenfunctions. We discuss only the continuous-time case, but analogous discrete-time results hold also.

2. NOTATION

A complex matrix-valued impulse response is easily motivated by a multiple-antenna wireless link having M transmit antennas and N receive antennas. Over a time interval $[T_{i0}, T_{i1}]$, an input signal, denoted $\bar{s}(\tau)$, an $M \times 1$ vector function of input time τ, drives the M transmit antennas. The resulting output at the N receive antennas, denoted $\bar{x}(t)$, is an $N \times 1$ vector function of output time t, where

$$\bar{x}(t) = \int_{T_{i0}}^{T_{i1}} d\tau \, H(t-\tau, \tau) \, \bar{s}(\tau) \,, \tag{1}$$

where $H(t-\tau, \tau)$ is an $N \times M$ matrix whose nm-th element is the impulse response, $h_{nm}(t-\tau, \tau)$, that connects the m-th transmit antenna to the n-th receive antenna, where the impulse is applied at time τ. Any linear time-varying, delay-spread channel is described by such an impulse response. For a time-invariant channel the dependence of $H(t-\tau, \tau)$ on its second variable vanishes.

3. SVD OF MATRIX IMPULSE RESPONSE

Let $H(t-\tau, \tau)$ be a $N \times M$ time-varying impulse response. Then for any specified finite input interval $[T_{i0}, T_{i1}]$ and any specified output interval $[T_{o0}, T_{o1}]$, either finite or infinite, the impulse response has the following exact representation over the indicated ranges of input time, τ, and output time, t,

$$\begin{aligned} H(t-\tau, \tau) \quad &= \quad \sum_{\ell=1}^{\infty} \bar{\phi}_\ell(t) v_\ell \bar{\psi}_\ell^\dagger(\tau), \\ & \qquad \tau \in [T_{i0}, T_{i1}], \; t \in [T_{o0}, T_{o1}], \end{aligned} \tag{2}$$

where $\{v_\ell\}$, the *singular values*, are nonnegative real scalars, $\{\bar{\phi}_\ell(t)\}$, the *output eigenfunctions*, are complex $N \times 1$ functions of t that are orthonormal over the output interval, and $\{\bar{\psi}_\ell(\tau)\}$, the *input eigenfunctions*, are complex $M \times 1$ functions of τ that are orthonormal over the input interval,

$$\int_{T_{o0}}^{T_{o1}} dt\, \bar{\phi}_j^\dagger(t)\, \bar{\phi}_\ell(t) = \delta_{jl}, \;\; j = 1, 2, \cdots, \;\; \ell = 1, 2, \cdots, \tag{3}$$

$$\int_{T_{i0}}^{T_{i1}} d\tau\, \bar{\psi}_j^\dagger(\tau)\, \bar{\psi}_\ell(\tau) = \delta_{jl}, \;\; j = 1, 2, \cdots, \;\; \ell = 1, 2, \cdots. \tag{4}$$

Note that the input and the output eigenfunctions are not eigenfunctions of the impulse response, but rather eigenfunctions of certain covariance kernels. In general a specification of *different* input or output intervals results in a *different* svd representation, in the same way that removing rows or columns from a matrix changes its svd.

The svd simultaneously diagonalizes the channel over both space and time. For a given input, $\bar{s}(\tau)$, the output of the channel (1) takes the form

$$\bar{x}(t) = \int_{T_{i0}}^{T_{i1}} d\tau \sum_{\ell: v_\ell > 0} \bar{\phi}_\ell(t) v_\ell \bar{\psi}_\ell^\dagger(\tau)\, \bar{s}(\tau), \quad t \in [T_{o0}, T_{o1}]. \tag{5}$$

After multiplying both sides of the expression by $\bar{\phi}_j^\dagger(t)$ (on the left), and integrating with respect to t, we obtain

$$x_j = v_j\, s_j, \;\; j = 1, 2, \cdots, \tag{6}$$

where the $\{s_j\}$ and $\{x_j\}$ are the scalar coefficients that represent the input signal and the output signal with respect to the orthonormal $\{\bar{\psi}_j(\tau)\}$ and $\{\bar{\phi}_j(t)\}$ respectively,

$$s_j = \int_{T_{i0}}^{T_{i1}} d\tau\, \bar{\psi}_j^\dagger(\tau) \bar{s}(\tau), \quad x_j = \int_{T_{o0}}^{T_{o1}} dt\, \bar{\phi}_j^\dagger(t) \bar{x}(t). \tag{7}$$

The attainable values of the output $\bar{x}(t)$ are linear combinations of those $\{\bar{\phi}_j(t)\}$ whose associated singular values are significantly greater than zero.

The power of the svd is immediately appreciated by considering the problem of choosing a finite duration input function $\bar{s}(\tau)$, subject to an energy constraint, to approximate a desired output function $\bar{d}(t)$, over a specified interval, in the mean-square sense. To do so we expand the input and output functions in terms of the input and output eigenfunctions respectively, and formulate the following minimization problem,

$$\min_{\{s_j\}} \left\{ \sum_{j=1}^{\infty} (d_j - v_j\, s_j)^2 + \lambda \sum_{j=1}^{\infty} s_j^2 \right\}, \tag{8}$$

where λ is a Lagrange multiplier. The optimum solution for the coefficients of the input waveform is

$$s_j = \frac{v_j\, d_j}{\lambda + v_j^2}, \; j = 1, 2, \cdots, \tag{9}$$

where λ is adjusted to satisfy the specified energy constraint. In contrast, this problem could not be tackled directly with Fourier methods because of the finite-time support of the input waveform.

4. DERIVATION OF SVD

We begin with the matrix version of Mercer's theorem [1, 5]. Starting with the $N \times M$ channel impulse response, we define a $M \times M$ Hermitian nonnegative-definite covariance kernel, $K(u, \tau)$, as follows,

$$K(u, \tau) = \int_{T_{o0}}^{T_{o1}} dt\, H^\dagger(t - u, u)\, H(t - \tau, \tau). \tag{10}$$

Under reasonable mathematical conditions, Kelly and Root [1] proved that the kernel can be expanded in terms of *vector* eigenfunctions and *scalar* eigenvalues as follows,

$$K(u, \tau) = \sum_{\ell=1}^{\infty} \bar{\psi}_\ell(u)\, v_\ell^2\, \bar{\psi}_\ell^\dagger(\tau), \quad u \in [T_{i0}, T_{i1}], \;\; \tau \in [T_{i0}, T_{i1}], \tag{11}$$

where

$$\int_{T_{i0}}^{T_{i1}} d\tau\, K(u, \tau)\, \bar{\psi}_\ell(\tau) = v_\ell^2\, \bar{\psi}_\ell(u), \quad u \in [T_{i0}, T_{i1}], \;\; \ell = 1, 2, \cdots, \tag{12}$$

the $\{v_\ell^2\}$ are nonnegative real eigenvalues, and the $\{\bar{\psi}_\ell(\tau)\}$ are $M \times 1$ orthonormal eigenfunctions that satisfy (4).

Next we use the vector-valued eigenfunctions $\{\bar{\psi}_\ell(\tau)\}$ as inputs to the channel to obtain $N \times 1$ vector-valued scaled output functions $\{\bar{\phi}_\ell(t)\}$. First, consider the case where the eigenvalue, v_ℓ^2, is strictly positive. Then we *define* the corresponding output function as follows,

$$\bar{\phi}_\ell(t) \equiv \frac{1}{v_\ell} \int_{T_{i0}}^{T_{i1}} d\tau\, H(t - \tau, \tau)\, \bar{\psi}_\ell(\tau). \tag{13}$$

A successive application of the definition (13), the definition of the covariance kernel (10), the eigenfunction/eigenvalue relation (12), and the orthogonality of the eigenfunctions (4) proves that the $\{\bar{\phi}_\ell(t)\}$ that are associated with strictly positive eigenvalues are mutually orthonormal,

$$\int_{T_{o0}}^{T_{o1}} dt\, \bar{\phi}_j^\dagger(t)\, \bar{\phi}_\ell(t) = \delta_{j\ell}, \;\; \forall\, j, \ell: \;\; v_j > 0, \; v_\ell > 0. \tag{14}$$

The output of the channel resulting from an application of an eigenfunction to the input of the channel whose eigenvalue is zero is equal to zero, because otherwise, a "squaring" and integration of the expression with respect to t would imply that the eigenvalue is greater than zero - a contradiction. For strictly positive eigenvalues we rewrite (13) as

$$\bar{\phi}_\ell(t)\, v_\ell = \int_{T_{i0}}^{T_{i1}} du\, H(t-u,u)\, \bar{\psi}_\ell(u)$$
$$t \in [T_{o0}, T_{o1}], \quad \ell = 1, 2, \cdots . \qquad (15)$$

Now we multiply both sides of (15) (on the right) by $\bar{\psi}_\ell^\dagger(\tau)$, and sum over ℓ such that v_ℓ is positive,

$$\begin{aligned} & \sum_{\ell: v_\ell > 0} \bar{\phi}_\ell(t) v_\ell \bar{\psi}_\ell^\dagger(\tau) \\ = & \int_{T_{i0}}^{T_{i1}} du H(t-u,u) \sum_{\ell: v_\ell > 0} \bar{\psi}_\ell(u) \bar{\psi}_\ell^\dagger(\tau), \\ & t \in [T_{o0}, T_{o1}], \;\; \tau \in [T_{i0}, T_{i1}] . \end{aligned} \qquad (16)$$

We claim that

$$\int_{T_{i0}}^{T_{i1}} du H(t-u,u) \sum_{\ell: v_\ell > 0} \bar{\psi}_\ell(u) \bar{\psi}_\ell^\dagger(\tau) = H(t-\tau, \tau) . \qquad (17)$$

To show this we apply an input to the channel which can be expressed as $\bar{s}(u) = \bar{s}_\|(u) + \bar{s}_\perp(u)$, where $\bar{s}_\|(u)$ is spanned by the eigenfunctions having positive eigenvalues, and where $\bar{s}_\perp(u)$ is orthogonal to those same eigenfunctions,

$$\bar{s}_\|(u) = \int_{T_{i0}}^{T_{i1}} d\tau \sum_{\ell: v_\ell > 0} \bar{\psi}_\ell(u) \bar{\psi}_\ell^\dagger(\tau) \bar{s}(\tau) , \qquad (18)$$

and $\bar{s}_\perp(u) = \bar{s}(u) - \bar{s}_\|(u)$. Of the two additive components of $\bar{s}(\cdot)$, the channel responds only to $\bar{s}_\|(\cdot)$. (It can be shown that the energy at the output of the channel due to $\bar{s}_\perp(\cdot)$ is equal to zero.) The channel response to $\bar{s}(\cdot)$ is identical to the response to $\bar{s}_\|(\cdot)$, so

$$\begin{aligned} & \int_{T_{i0}}^{T_{i1}} d\tau H(t-\tau,\tau) \bar{s}(\tau) \\ = & \int_{T_{i0}}^{T_{i1}} du H(t-u,u) \int_{T_{i0}}^{T_{i1}} d\tau \sum_{\ell: v_\ell > 0} \bar{\psi}_\ell(u) \bar{\psi}_\ell^\dagger(\tau) \bar{s}(\tau) . \end{aligned}$$

This proves (17) which, when substituted into (16) yields the desired svd representation.

5. SPECIAL CASES OF SVD

We now consider some special cases of the svd: the diagonal kernel, an impulse response that is separable in space and time, the time-invariant channel for large time-bandwidth product, and finally the ideal bandlimited scalar channel. For three of these cases the svd exhibits distinct space-time structure. In general the svd makes no distinction between spatial and temporal degrees of freedom, and one cannot associate individual singular values with either spatial or temporal diversity.

5.1. Diagonal kernel

Here the $M \times M$ kernel (10) is diagonal, and the m-th diagonal element is itself a scalar kernel, $k_m(u, \tau)$. In turn, each scalar kernel has an expansion in terms of eigenvalues and scalar eigenfunctions. The corresponding vector-valued eigenfunctions of the matrix kernel have all but one of their M components equal to zero. Thus the svd has a natural double-index form,

$$H(t-\tau,\tau) = \sum_{m=1}^{M} \sum_{q=1}^{\infty} \bar{\phi}_{mq}(t)\, v_{mq}\, \bar{\psi}_{mq}^\dagger(\tau) ,$$
$$\tau \in [T_{i0}, T_{i1}], \;\; t \in [T_{o0}, T_{o1}] , \qquad (19)$$

where the m-index denotes space, the q-index denotes time, and only the m-th component of $\bar{\psi}_{mq}(\tau)$ is nonzero, where

$$\int_{T_{i0}}^{T_{i1}} d\tau\, k_m(u,\tau) [\bar{\psi}_{mq}(\tau)]_m = v_{mq}^2\, [\bar{\psi}_{mq}(u)]_m ,$$
$$u \in [T_{i0}, T_{i1}], \;\; m = 1, \cdots, M , \;\; q = 1, 2, \cdots . (20)$$

5.2. Separable impulse response

A separable impulse response, characteristic of the flat-fading multiple-antenna channel, is equal to the product of a constant matrix, G, and a scalar impulse response, $f(t-\tau, \tau)$,

$$H(t-\tau,\tau) = G\, f(t-\tau,\tau) . \qquad (21)$$

In turn, each of the two factors has its own svd,

$$\begin{aligned} G &= \sum_{m=1}^{M} \bar{\alpha}_m\, \mu_m \bar{\beta}_m^\dagger , \\ f(t-\tau,\tau) &= \sum_{q=1}^{\infty} \theta_q(t)\, z_q\, \xi_q(\tau)^* . \end{aligned}$$

Again we have a double-index (e.g., space-time) singular value decomposition of the form (19), where

$$\begin{aligned} \bar{\phi}_{mq}(t) &= \bar{\alpha}_m\, \theta_q(t) , \\ v_{mq} &= \mu_m\, z_q , \\ \bar{\psi}_{mq}(\tau) &= \bar{\beta}_m\, \xi_q(\tau) . \end{aligned}$$

5.3. Time-invariant channel, large time-bandwidth product

Here the channel impulse response is $H(t-\tau,\tau) = H(t-\tau)$, whose matrix-valued frequency response is

$$\hat{H}(f) = \int du\, H(u)\, e^{-i2\pi f u} .$$

We specify the same input and output intervals of $[-T/2, T/2]$, and we assume that the frequency response is smooth over frequency intervals of duration comparable to $1/T$. Then asymptotically, for large time-bandwidth products, $TB \gg 1$, where B is the channel bandwidth, the channel impulse response again has a double-index svd,

$$H(t-\tau,\tau) \approx \sum_{m=1}^{M} \sum_{q=1}^{\infty} \bar{\phi}_{mq}(t)\, v_{mq}\, \bar{\psi}^{\dagger}_{mq}(\tau)\,,$$
$$\tau \in [-T/2, T/2],\ \ t \in [-T/2, T/2]\,, \tag{22}$$

where

$$\begin{aligned} \bar{\phi}_{mq}(t) &= \bar{\alpha}_{mq}\, e^{i2\pi qt/T}/\sqrt{T}\,, \\ \bar{\psi}_{mq}(\tau) &= \bar{\beta}_{mq}\, e^{i2\pi q\tau/T}/\sqrt{T} \end{aligned}$$

where the $\{\bar{\alpha}_{mq}\}$, $\{\bar{\beta}_{mq}\}$, and $\{v_{mq}\}$ constitute the svd of the matrix-valued frequency response, evaluated at the frequency $f = q/T$,

$$\hat{H}(q/T) = \sum_{m=1}^{M} \bar{\alpha}_{mq}\, v_{mq}\, \bar{\beta}^{\dagger}_{mq}\,. \tag{23}$$

Multiple-antenna OFDM (orthogonal frequency division multiplex) techniques can be interpreted in light of the svd (22). The wideband complex pulse that is fed to each transmit antenna is a linear combination of harmonically related sinewaves, that are extended periodically (the *cyclic prefix*) beyond the interval $[-T/2, T/2]$ by an amount that exceeds the duration of the channel impulse response. This ensures that all of the output sinewaves exist in a transient-free state over an interval of duration greater than T.

5.4. Ideal bandlimited scalar channel

The ideal bandlimited scalar channel has an impulse response

$$\begin{aligned} h(t-\tau) &= \int_{-B/2}^{B/2} df\, e^{i2\pi f(t-\tau)} \\ &= \frac{\sin(\pi B(t-\tau))}{\pi(t-\tau)}\,. \end{aligned} \tag{24}$$

The svd of the channel, for the input interval $[-T/2, T/2]$, and the output interval $(-\infty, \infty)$ is

$$h(t-\tau) = \sum_{\ell=1}^{\infty} \phi_\ell(t)\, v_\ell\, \psi_\ell(\tau)\,,$$
$$t \in (-\infty, \infty),\ \ \tau \in [-T/2, T/2]\,. \tag{25}$$

The $\{\phi_\ell(t)\}$ are called *prolate spheroidal wave functions*.

The covariance kernel for which the $\{\psi_\ell(\tau)\}$ are eigenfunctions is equal to the channel impulse response itself, a property unique to this channel. This property implies that the output eigenfunctions, in addition to being orthonormal over the infinite time interval, are also orthogonal over the finite interval $[-T/2, T/2]$. To see this, we proceed as follows

$$\begin{aligned} v_\ell\, \phi_\ell(t) &= \int_{-T/2}^{T/2} d\tau\, h(t-\tau)\, \psi_\ell(\tau)\,,\ \ t \in (-\infty, \infty) \\ &= \int_{-T/2}^{T/2} d\tau\, k(t-\tau)\, \psi_\ell(\tau) \\ &= v_\ell^2\, \psi_\ell(t)\,,\ \ t \in [-T/2, T/2]\,. \end{aligned} \tag{26}$$

As a result of equating the first and last terms of the above equations, we have

$$\phi_\ell(t) = v_\ell\, \psi_\ell(t),\ \ t \in [-T/2, T/2]\,,\ \ \ell = 1, 2, \cdots\,.$$

The double-orthogonality of the $\phi_\ell(t)$ implies that the svd representation (25), where the output interval is $(-\infty, \infty)$, yields a second svd representation where the output interval is $[-T/2, T/2]$,

$$h(t-\tau) = \sum_{\ell=1}^{\infty} \psi_\ell(t)\, v_\ell^2\, \psi_\ell(\tau)\,,$$
$$t \in [-T/2, T/2],\ \ \tau \in [-T/2, T/2]\,. \tag{27}$$

6. CONCLUSIONS

The svd of the matrix-valued impulse response is a powerful tool that enables us to analyze waveform channels with the same facility that we analyze ordinary matrix operations.

7. REFERENCES

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