AN EFFICIENT ALGORITHM FOR SOLVING THE DOWNLINK BEAMFORMING PROBLEM WITH INDEFINITE CONSTRAINTS

David Samuelsson, Mats Bengtsson and Björn Ottersten*

Dept. Signals, Sensors & Systems, KTH
SE-100 44 Stockholm, Sweden

ABSTRACT

By imposing additional constraints in the downlink beamforming optimization, more general Quality of Service (QoS) measures than the average Signal to Interference and Noise Ratio (SINR) alone, can be introduced.

Herein a rapidly converging algorithm solving the downlink beamforming problem with additional indefinite quadratic constraints on the beamforming vector is presented. The proposed algorithm is significantly faster than the previously proposed solution, which involves semidefinite programming. Also, the algorithm is easy to implement, since it only involves eigenvalue problems.

1. INTRODUCTION

By deploying antenna arrays at the transmitter, beamforming techniques can be used to exploit spatial channel knowledge to increase spectral efficiency and downlink capacity.

The objective of the downlink beamforming problem is to find beamforming vectors and allocate power to ensure that each user achieves a targeted QoS. The average SINR is commonly taken as the QoS measure. There are however many cases in which the QoS measure should take other factors than average SINR into consideration, which motivates the use of additional constraints.

The unconstrained downlink beamforming problem was first solved in [1]. A fast algorithm, also coping with infeasible scenarios, was later developed in [2]. A conceptually different semidefinite relaxation approach was presented in [3].

In [4], additional indefinite constraints on the beamformer were used to ensure a minimum level of multi-path diversity in CDMA systems. An algorithm, based on semidefinite programming, was developed to solve the constrained downlink beamforming problem. Other applications of indefinite constraints include limiting interference in certain spatial directions.

Herein the algorithm proposed in [2] is modified to allow indefinite constraints on the beamformer. The proposed algorithm is considerably faster than the semidefinite approach suggested in [4]. The implementation of the algorithm proposed herein only involves solving eigenvalue problems, contrary to the previously known algorithm, which involves semidefinite programming.

1.1. Signal model

In this work a downlink single cell scenario is considered. The base station is assumed to be equipped with an M element array of antennas. The signal, $s_i(t)$, intended for user i is mapped onto the array by the beamforming vector, $\mathbf{u}_i \in \mathbb{C}^M$, which throughout this work is assumed normalized to $\|\mathbf{u}_i\|_2 = 1$.

The signal received by user i, is modeled as

$$r_i(t) = \sum_{k=1}^{K} \mathbf{h}_i^*(t)\mathbf{u}_k s_k(t) + n_i(t),$$

where $\mathbf{h}_i(t) \in \mathbb{C}^M$ is the time-varying spatial vector channel to user i, n_i is additive white noise with variance σ_i^2 and $\{\cdot\}^*$ denote Hermitian conjugate transpose. It is assumed that $\mathbf{h}_i(t)$, $s_j(t)$ and $n_k(t)$ $(\forall i, j, k)$ are uncorrelated processes.

Since $\mathbf{u}_i$ is normalized, the downlink transmission powers are given by $p_i = \mathrm{E}\left\{|s_i(t)|^2\right\}$, which are stacked in the vector $\mathbf{p}$. The total transmission power is thus given by $\|\mathbf{p}\|_1$. For notational convenience, the beamformers are also collected in the matrix $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_K]$.

By defining the spatial channel covariance matrices as

$$\mathbf{R}_i \triangleq \mathrm{E}\left\{\mathbf{h}_i(t)\mathbf{h}_i^*(t)\right\}, \quad 1 \leq i \leq K,$$

the downlink SINR at user i is given by

$$\mathrm{SINR}_i^{\mathrm{DL}}(\mathbf{U}, \mathbf{p}) = \frac{p_i \mathbf{u}_i^* \widetilde{\mathbf{R}}_i \mathbf{u}_i}{\sum_{k \neq i} p_k \mathbf{u}_k^* \widetilde{\mathbf{R}}_i \mathbf{u}_k + 1}, \tag{1}$$

where $\widetilde{\mathbf{R}}_i$ is the scaled covariance matrix $\widetilde{\mathbf{R}}_i = \mathbf{R}_i / \sigma_i^2$.

1.2. Problem formulation

The traditional objective of downlink beamforming is to ensure that each user, i, achieves a target individual SINR, γ_i. In this work more general constraints are considered, where in addition to the SINR targets, the beamforming vectors are required to satisfy an indefinite quadratic constraint of the form,

$$\mathbf{u}_i^* \mathbf{C}_i \mathbf{u}_i \geq 0, \ \forall i. \tag{2}$$

The typically indefinite matrices, $\mathbf{C}_i$, can be chosen arbitrarily to formulate many types of constraints, as mentioned above.

For notational convenience, the set $\mathcal{C} \triangleq \{\mathbf{U} | \mathbf{u}_i^* \mathbf{C}_i \mathbf{u}_i \geq 0 \ \forall i\}$ is defined to denote matrices $\mathbf{U}$ that satisfy the indefinite constraints (2).

The SINR constraints can be formulated in many different ways. A convenient measure is the minimum, normalized, downlink SINR, which is defined as

$$\mathrm{SINR}_{\min}^{\mathrm{DL}}(\mathbf{U}, \mathbf{p}) \triangleq \min_{1 \leq i \leq K} \frac{\mathrm{SINR}_i^{\mathrm{DL}}(\mathbf{U}, \mathbf{p})}{\gamma_i}. \tag{3}$$

*david.samuelsson@s3.kth.se

Hence $\mathrm{SINR}^{\mathrm{DL}}_{\min}(\mathbf{U},\mathbf{p}) \geq 1$ corresponds to all SINR targets being met.

Two different problem formulations are considered in this work. The first formulation is useful, primarily to verify that the targeted SINRs are feasible.

P 1. *maximize* $\mathrm{SINR}^{\mathrm{DL}}_{\min}(\mathbf{U},\mathbf{p})$ *subject to* $\|\mathbf{p}\|_1 \leq P_{\max}$.

Problem (P1) maximizes the performance of the "weakest" user, given a maximum transmission power, $\mathrm{P}_{\max}$. The optimal value of $\mathrm{SINR}^{\mathrm{DL}}_{\min}$ is a measure of the spatial separability of the users. If the optimal $\mathrm{SINR}^{\mathrm{DL}}_{\min} \geq 1$, the SINR targets are feasible, otherwise actions should be taken to relax the conditions.

At the optimum, the SINRs are balanced [2],

$$\mathrm{SINR}^{\mathrm{DL}}_{\min}(\mathbf{U}_{\text{opt}},\mathbf{p}_{\text{opt}}) = \frac{\mathrm{SINR}^{\mathrm{DL}}_{i}(\mathbf{U}_{\text{opt}},\mathbf{p}_{\text{opt}})}{\gamma_i},\ 1 \leq i \leq K, \quad (4)$$

and Problem (P1) is therefore referred to as the SINR balancing problem.

For feasible scenarios, the second problem formulation (P2) is of great interest, since it minimizes the inter-cell interference without violating the SINR targets.

P 2. *Minimize* $\|\mathbf{p}\|_1$ *subject to* $\mathrm{SINR}^{\mathrm{DL}}_{min}(\mathbf{U},\mathbf{p}) \geq 1$.

Problem (P2) minimizes the total transmission power, and is therefore referred to as the power minimization problem.

The two optimization problems are closely related. They are in fact equivalent if $\mathrm{P}_{\max}$ is chosen as the optimal minimum power of Problem (P2). The main focus of this work is to provide an efficient algorithmic solution to the two beamforming problems, (P1) and (P2), with the beamformers satisfying the non-trivial non-convex constraint $\mathbf{U} \in \mathcal{C}$.

2. SOLVING THE UNCONSTRAINED BEAMFORMING PROBLEMS

The proposed algorithm, solving the constrained beamforming problem, is closely related to the algorithm proposed in [2], for the unconstrained case. In this section, the solution to the unconstrained beamforming problem is presented, providing the basic framework for imposing indefinite constraints in Section 3.

By introducing the matrix

$$\mathbf{D}(\mathbf{U}) = \operatorname{diag}\left\{\frac{\gamma_1}{\mathbf{u}_1^*\widetilde{\mathbf{R}}_i\mathbf{u}_1},\ldots,\frac{\gamma_K}{\mathbf{u}_K^*\widetilde{\mathbf{R}}_K\mathbf{u}_K}\right\}$$

and the cross-talk matrix

$$[\mathbf{\Psi}(\mathbf{U})]_{ik} = \begin{cases} \mathbf{u}_k^*\widetilde{\mathbf{R}}_i\mathbf{u}_k, & k \neq i \\ 0, & k = i \end{cases},$$

the SINR constraints are given in matrix form by

$$\big(\mathbf{I} - \mathbf{D}\mathbf{\Psi}(\mathbf{U})\big)\mathbf{p} \geq \mathbf{D}\mathbf{1}, \quad (5)$$

where $\geq$ is taken element-wise. For optimal $\mathbf{U}$ and $\mathbf{p}$, Inequality (5) is satisfied with equality, in accordance with Equation (4), and $\mathbf{p}_{\text{opt}}$ can therefore be obtained from $\mathbf{U}_{\text{opt}}$ as

$$\mathbf{p}_{\text{opt}} = \big(\mathbf{I} - \mathbf{D}\mathbf{\Psi}(\mathbf{U}_{\text{opt}})\big)^{-1}\mathbf{D}\mathbf{1}. \quad (6)$$

2.1. Virtual uplink-downlink duality

There is an interesting duality between uplink and downlink problems, which motivates the introduction of the, so called, virtual uplink SINR of user i, $\mathrm{SINR}^{\mathrm{VUL}}_i$.

$$\mathrm{SINR}^{\mathrm{VUL}}_i(\mathbf{u}_i,\mathbf{q}) \triangleq \frac{q_i\mathbf{u}_i^*\widetilde{\mathbf{R}}_i\mathbf{u}_i}{\mathbf{u}_i^*(\sum_{k\neq i} q_k\widetilde{\mathbf{R}}_i + \mathbf{I})\mathbf{u}_i}, \quad (7)$$

where $\mathbf{q} = [q_1,\ldots,q_K]^{\mathrm{T}}$ is the virtual uplink power distribution. Note that the different $\mathrm{SINR}^{\mathrm{VUL}}$ for the users are only coupled through the uplink power distribution $\mathbf{q}$, whereas the optimization over $\mathbf{U}$ is decoupled. The SINR balancing Problem (P4) and the power minimization Problem (P2) with respect to the virtual uplink SINRs are therefore much simpler to analyze and to solve. Theorem 1, proved in [1] and [2], is therefore powerful since it reduces the complexity of the downlink problem to that of the uplink.

Theorem 1. *For given* $\mathbf{U}$ *and* $P_{\max}$ *it holds that*

$$\begin{aligned}\max_{\mathbf{p}}\min_{1\leq i\leq K}\mathrm{SINR}^{\mathrm{DL}}_i(\mathbf{U},\mathbf{p}) &= \\ = \max_{\mathbf{q}}\min_{1\leq i\leq K}&\mathrm{SINR}^{\mathrm{VUL}}_i(\mathbf{u}_i,\mathbf{q}),\end{aligned} \quad (8)$$

where the maximizations are constrained by $\|\mathbf{p}\|_1 \leq P_{\max}$ *and* $\|\mathbf{q}\|_1 \leq P_{\max}$*, respectively.*

It follows from Theorem 1 that the optimal virtual uplink beamformer $\mathbf{U}_{\text{opt}}$ is also optimal for the downlink, regardless of which of the Problems (P1) and (P2) is considered. The virtual uplink power distribution that optimally balances the virtual uplink $\mathrm{SINR}^{\mathrm{VUL}}_i$, is similarly to equation (5) given by

$$\mathbf{q}_{\text{opt}} = \big(\mathbf{I} - \mathbf{D}\mathbf{\Psi}^{\mathrm{T}}(\mathbf{U})\big)^{-1}\mathbf{D}\mathbf{1}. \quad (9)$$

Note that for feasible $\mathbf{U}$, the elements of $\mathbf{q}_{\text{opt}}$ are strictly positive and minimize $\|\mathbf{q}\|_1$ without violating the SINR constraints.

2.2. Algorithmic solutions for unconstrained beamformers

In [2] two algorithms are proposed, solving the unconstrained downlink SINR balancing Problem (P1) and the power minimization Problem (P2), respectively. It was shown that the optimization could be reduced to two sub-problems,

P 3. $\mathbf{q}_{opt}(\widetilde{\mathbf{U}},P_{\max}) = \arg\max_{\|\mathbf{q}\|_1\leq P_{\max}} \mathrm{SINR}^{\mathrm{VUL}}_{\min}(\widetilde{\mathbf{U}},\mathbf{q})$,

P 4. $\mathbf{U}_{opt}(\widetilde{\mathbf{q}}) = \arg\max_{\mathbf{U}} \mathrm{SINR}^{\mathrm{VUL}}_{\min}(\mathbf{U},\widetilde{\mathbf{q}})$

and that the global optimum of (P1) and (P2) are solved by the algorithms stated in Table 1, which rapidly converge to the global optimal values.

In [2], the solution $\mathbf{q}_{\text{opt}}$ of Problem (P3) is shown to be given by the eigenvalue problem

$$\mathbf{\Lambda}(\widetilde{\mathrm{U}},\mathrm{P}_{\max})\mathbf{q}_{\text{ext}} = \lambda_{\max}\mathbf{q}_{\text{ext}}, \quad (10)$$

where $\mathbf{q}_{\text{ext}} = \binom{\mathbf{q}_{\text{opt}}}{1}$ is the eigenvector associated with the largest eigenvalue $\lambda_{\max}$ of the extended uplink coupling matrix $\mathbf{\Lambda}(\widetilde{\mathrm{U}},\mathrm{P}_{\max})$, defined as

$$\mathbf{\Lambda}(\widetilde{\mathrm{U}},\mathrm{P}_{\max}) = \begin{bmatrix} \mathbf{D}\mathbf{\Psi}^{\mathrm{T}}(\mathrm{U}) & \mathbf{D}\mathbf{1} \\ \frac{1}{\mathrm{P}_{\max}}\mathbf{1}^{\mathrm{T}}\mathbf{D}\mathbf{\Psi}^{\mathrm{T}}(\mathrm{U}) & \frac{1}{\mathrm{P}_{\max}}\mathbf{1}^{\mathrm{T}}\mathbf{D}\mathbf{1} \end{bmatrix}. \quad (11)$$

Table 1. Algorithmic solutions to the SINR balancing Problem (P1) and the power minimization Problem (P2). Below $\Delta\{\cdot\}$ denotes the difference of a quantity in two consecutive iterations.

(a) Solution of (P1)	(b) Solution of (P2)
$\widetilde{\mathbf{q}} = \mathbf{0}$	$\widetilde{\mathbf{q}} = \mathbf{0}$
repeat	**repeat**
$\quad \widetilde{\mathbf{U}} = \mathbf{U}_{\text{opt}}(\widetilde{\mathbf{q}})$	$\quad \widetilde{\mathbf{U}} = \mathbf{U}_{\text{opt}}(\widetilde{\mathbf{q}})$
$\quad \widetilde{\mathbf{q}} = \mathbf{q}_{\text{opt}}(\widetilde{\mathbf{U}}, \mathrm{P}_{\max})$	$\quad \widetilde{\mathbf{q}} = \mathbf{q}_{\text{opt}}(\widetilde{\mathbf{U}}, \mathrm{P}_{\max})$
until $\Delta\, \mathrm{SINR}^{\mathrm{VUL}}_{\min} \leq \epsilon$	**until** $\mathrm{SINR}^{\mathrm{VUL}}_{\min} \geq 1$
$\widetilde{\mathbf{p}} = \left(\mathbf{I} - \mathbf{D}\boldsymbol{\Psi}(\widetilde{\mathbf{U}})\right)^{-1}\mathbf{D}\mathbf{1}$	**repeat**
	$\quad \widetilde{\mathbf{U}} = \mathbf{U}_{\text{opt}}(\widetilde{\mathbf{q}})$
	$\quad \widetilde{\mathbf{q}} = (\mathbf{I} - \mathbf{D}\boldsymbol{\Psi}^{\mathrm{T}}(\widetilde{\mathbf{U}}))^{-1}\mathbf{D}\mathbf{1}$
	until $\Delta\|\widetilde{\mathbf{q}}\|_1 \leq \epsilon$
	$\widetilde{\mathbf{p}} = \left(\mathbf{I} - \mathbf{D}\boldsymbol{\Psi}(\widetilde{\mathbf{U}})\right)^{-1}\mathbf{D}\mathbf{1}$

The solution to Problem (P4), derived in [2], is given by the K, decoupled problems

$$\mathbf{u}_i = \arg\max_{\mathbf{u}} \mathrm{SINR}^{\mathrm{VUL}}(\mathbf{u}_i, \widetilde{\mathbf{q}}) = \arg\max_{\mathbf{u}_i} \frac{\mathbf{u}_i^* \widetilde{\mathbf{R}}_i \mathbf{u}_i}{\mathbf{u}_i^* \mathbf{Q}_i \mathbf{u}_i}, \quad (12)$$

where $\mathbf{Q}_i = \sum_{k \neq i} q_k \widetilde{\mathbf{R}}_i + \mathbf{I}$. If $\mathbf{u}_i$ is unconstrained, this is a generalized eigenvalue problem and the solution is given by the generalized eigenvector associated with the largest generalized eigenvalue $\lambda_{\max}(\widetilde{\mathbf{R}}_i, \mathbf{Q}_i)$, such that $\widetilde{\mathbf{R}}_i \mathbf{u}_i = \lambda_{\max} \mathbf{Q}_i \mathbf{u}_i$.

3. CONSTRAINING THE BEAMFORMERS

The results of Section 2 and the proposed algorithms in Table 1 are unaffected by imposing the indefinite constraint $\mathbf{U} \in \mathcal{C}$. Problem (P4) is however severely complicated by the imposed indefinite non-convex constraints. The constrained version of problem (P 4) is given by

P 4C. $\quad \mathbf{u}_i = \arg\max_{\mathbf{u}_i \in \mathcal{P}_{\mathbf{C}_i}} \frac{\mathbf{u}_i^* \widetilde{\mathbf{R}}_i \mathbf{u}_i}{\mathbf{u}_i^* \mathbf{Q}_i \mathbf{u}_i}$,

where $\mathcal{P}_{\mathbf{C}_i} \triangleq \{\mathbf{u} | \mathbf{u}^* \mathbf{C}_i \mathbf{u} \geq 0\}$. This optimization problem is analyzed in Section 4 where an efficient algorithmic solution, is derived. Hence by solving the constrained problem (P4C) in place of the unconstrained, the algorithms in Table 1 will converge to the globally optimal $\mathbf{U} \in \mathcal{C}$.

4. SOLVING THE CONSTRAINED PROBLEM

In this section an efficient algorithmic solution to Problem (P4C) is derived. It is shown that the optimal solution can be obtained from the, readily solvable, dual problem. For notational convenience, all indices are dropped and the problem under study is

P 5. $\max_{\mathbf{u} \in \mathcal{P}_{\mathbf{C}}} \frac{\mathbf{u}^* \mathbf{R} \mathbf{u}}{\mathbf{u}^* \mathbf{Q} \mathbf{u}}$, *where* $\mathbf{Q} \succ \mathbf{0}$.

Here, $\mathbf{Q} \succ \mathbf{0}$, denotes a positive definite matrix $\mathbf{Q}$.

4.1. The dual problem

By noting that the criterion function is independent of the normalization of $\mathbf{u}$ and that $\mathbf{Q} \succ \mathbf{0}$, the constraint $\mathbf{u} \in \mathcal{P}$ can be replaced by $\frac{\mathbf{u}^* \mathbf{C} \mathbf{u}}{\mathbf{u}^* \mathbf{Q} \mathbf{u}} \geq 0$, which, see [5], results in the dual problem

P 6. $\min_{\nu \geq 0} \max_{\mathbf{u}} \left[\frac{\mathbf{u}^* (\mathbf{R} + \nu \mathbf{C}) \mathbf{u}}{\mathbf{u}^* \mathbf{Q} \mathbf{u}} \right]$.

For a given ν, the maximization over $\mathbf{u}$ is a generalized eigenvalue problem. The maximum is given by the largest generalized eigenvalue, $\lambda_{\max}(\mathbf{R} + \nu \mathbf{C}, \mathbf{Q})$ and the maximum is attained for the associated generalized eigenvector $\mathbf{u}_{\text{opt}}$. The dual problem can therefore equivalently be written as

$$\min_{\nu \geq 0} \lambda_{\max}(\mathbf{R} + \nu \mathbf{C}, \mathbf{Q}).$$

4.2. Relation between the primal and dual problem

Below, the relation between the dual and primal problem is discussed. Any dual problem provides an upper bound on the associated primal problem, by week duality [5]. The main result of this section is Theorem 2, which states that the dual Problem (P6) and the primal Problem (P5) do in fact attain the same optimal value.

Theorem 2. *The primal Problem (P5) and the dual Problem (P6) have the same optimal value.*

Proof. The strong duality is proven by showing that there is an optimal point $(\nu_{\text{opt}}, \mathbf{u}_{\text{opt}})$ of the dual problem satisfying

$$\nu_{\text{opt}} \mathbf{u}_{\text{opt}}^* \mathbf{C} \mathbf{u}_{\text{opt}} = 0 \quad (13)$$

$$\mathbf{u}_{\text{opt}}^* \mathbf{C} \mathbf{u}_{\text{opt}} \geq 0 \quad (14)$$

which proves that $\mathbf{u}_{\text{opt}}$ is in the feasible set of the primal problem and that the primal and dual criterion functions attain the same value at this point.

Case 1: $\nu_{\text{opt}} = 0$
This implies that there exists a $\mathbf{u}_{\text{opt}}$ in the eigenspace associated with $\lambda_{\max}$ such that $\mathbf{u}_{\text{opt}}^* \mathbf{C} \mathbf{u}_{\text{opt}} \geq 0$, since otherwise $\lambda_{\max}(\mathbf{R} + \nu_{\text{opt}} \mathbf{C}, \mathbf{Q})$ could be decreased by increasing ν. $(\nu_{\text{opt}}, \mathbf{u}_{\text{opt}})$ thereby satisfies conditions (13) and (14).

Case 2: $\nu_{\text{opt}} > 0$
By the same argument as in case 1, there exist a $\mathbf{u}_p$ in the eigenspace associated with $\lambda_{\max}$, such that $\mathbf{u}_p^* \mathbf{C} \mathbf{u}_p \geq 0$.

Similarly there must exist a $\mathbf{u}_n$ in the eigenspace of $\lambda_{\max}$, such that $\mathbf{u}_n^* \mathbf{C} \mathbf{u}_n \leq 0$, since otherwise $\lambda_{\max}(\mathbf{R} + \nu_{\text{opt}} \mathbf{C}, \mathbf{Q})$ could be decreased by decreasing ν.

Let $\mathbf{u}(\alpha) = \alpha \mathbf{u}_p + (1 - \alpha) \mathbf{u}_n$ and note that $\mathbf{u}(\alpha)$ also is an eigenvector associated with $\lambda_{\max}$. Define the real continuous function $f(\alpha) = \mathbf{u}(\alpha)^* \mathbf{C} \mathbf{u}(\alpha)$. It holds that $f(0) \leq 0 \leq f(1)$ and thus there exist an $\alpha_0 \in [0, 1]$ such that $f(\alpha_0) = \mathbf{u}(\alpha_0)^* \mathbf{C} \mathbf{u}(\alpha_0) = 0$. It follows that $\mathbf{u}_{\text{opt}} = \mathbf{u}(\alpha_0)$ satisfies conditions (13) and (14).

□

The next corollary is an immediate consequence of Theorem 2 and the proof thereof, and is the key to solving Problem (P5).

Corollary 3. *Any optimal point* $\mathbf{u}_{opt}$ *of Problem (P5) is in the generalized eigenspace associated with* $\lambda_{\max}(\mathbf{R} + \nu_{opt} \mathbf{C}, \mathbf{Q})$*, where* ν_{opt} *is the optimal point of the convex problem*

$$\nu_{opt} = \arg\min_{\nu \geq 0} \lambda_{\max}(\mathbf{R} + \nu \mathbf{C}, \mathbf{Q}).$$

Conversely, any normalized $\mathbf{u}$ *in the generalized eigenspace of* $\lambda_{\max}(\mathbf{R} + \nu_{opt} \mathbf{C}, \mathbf{Q})$*, satisfying condition (13) and (14) is an optimal solution of Problem (P5). Furthermore, there exists at least one such point.*

4.3. Solving the dual problem

The dual problem is always convex, see [5], in the dual variable, ν. It can therefore readily be solved using virtually any standard method for one dimensional line searches. However, the criterion function has several nice properties that allow for an efficient line search. For example, it can be shown that the derivative in each point is given by

$$\frac{\partial}{\partial \nu}\lambda_{\max}(\mathbf{R}+\nu\mathbf{C},\mathbf{Q}) = \frac{\mathbf{u}^*\mathbf{C}\mathbf{u}}{\mathbf{u}^*\mathbf{Q}\mathbf{u}}, \tag{15}$$

where $\mathbf{u}$ is the eigenvector associated with the maximum eigenvalue. An efficient implementation converges in typically less than ten iterations, see Section 5.

4.4. Solving the primal problem

In Section 4.3 an efficient algorithm to find the optimal ν_{opt} of the dual Problem (P6) is presented. The optimal $\mathbf{u}_{\text{opt}}$ of the primal Problem (P5) is thus, according to Corollary 3, confined to the eigenspace associated with $\lambda_{\text{opt}} = \lambda_{\max}(\mathbf{R}+\nu_{\text{opt}}\mathbf{C},\mathbf{Q})$. The eigenspace associated with λ_{opt} is typically of dimension one and any vector in the eigenspace is thus optimal and can be chosen as $\mathbf{u}_{\text{opt}}$.

A higher dimensionality eigenspace corresponds to perfect alignment of some of the generalized eigenvectors of $\mathbf{R}$ and $\mathbf{C}$. An optimal $\mathbf{u}_{\text{opt}}$ can then be obtained in several ways. The most straightforward way is to remove the symmetry by adding small random permutations to the matrices, which yield an eigenspace of dimension one.

A more brute force approach is to find a $\mathbf{u}_{\text{opt}}$ in the eigenspace of λ_{opt} that satisfies condition (13) and (14) which, according to Corollary 3, solves the primal problem. Such a $\mathbf{u}_{\text{opt}}$ can be computed using a similar approach as in the proof of Theorem 2.

5. COMPUTATIONAL COMPLEXITY

The complexity of the proposed algorithm for constrained beamformers, is closely related to the algorithm for arbitrary beamformers of [2], which is described in Section 2.

A well implemented line-search of the constrained Problem (P4C) converges in typically less than 10 iterations, if the constraint is active, and on the first iteration for inactive constraints. Each iteration involves a generalized eigenvalue problem. The complexity is thus between a factor one and 10 of that of the unconstrained version.

In Figure 1, the processing time of the proposed algorithm is illustrated and compared to other algorithms. Random scenarios were generated and the indefinite constraints were designed to be active with probability 50%.

It can be observed from the figure that the processing time of the proposed algorithm is increased by a factor 5 compared to the unconstrained algorithm. The processing time of the algorithm based on the semidefinite approach is approximately a factor 5 higher than the proposed algorithm, for these problem sizes.

6. CONCLUSIONS

In this work an efficient algorithm solving a constrained downlink beamforming problem has been proposed. The algorithm is based on a simpler algorithm for the unconstrained case, which was modified to cope with beamformers limited by an indefinite constraint.

The proposed algorithm was shown to converge considerably faster than any other known algorithm for the considered problem. Furthermore, the implementation only involves eigenvalue problems, which are readily implemented, contrary to other techniques which involve semidefinite programming.

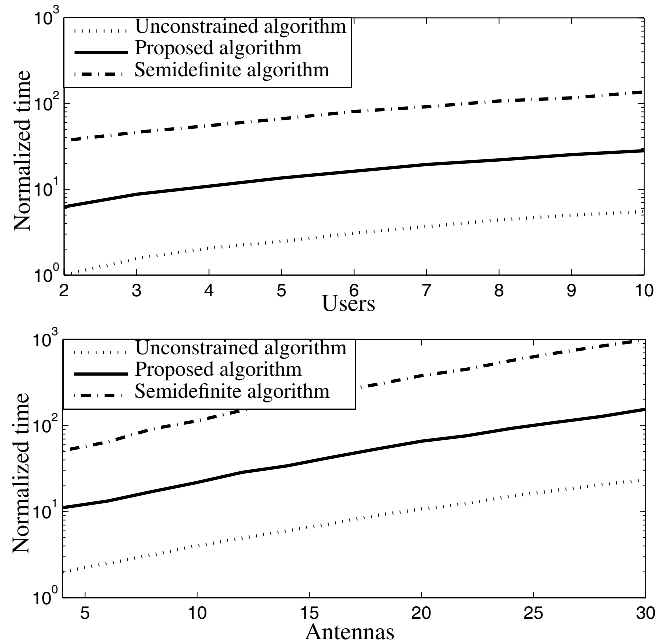

Fig. 1. The normalized average processing time is plotted as a function of the number of users (top figure) and antennas (bottom figure) for the unconstrained algorithm of [2], the algorithm proposed herein and the semidefinite approach of [4], respectively.

7. REFERENCES

[1] Farrokh Rashid-Farrokhi, K.J. Ray Liu, and Leandros Tassiulas, "Transmit beamforming and power control for cellular wireless systems," *IEEE Journal on Selecter Areas of Communications*, vol. 16, no. 8, pp. 1437–1450, Oct. 1998.

[2] Martin Schubert and Holger Boche, "Solution of the multiuser downlink beamforming problem with individual SINR constraints," *IEEE Trans. Vehicular Technology*, vol. 53, no. 1, January 2004.

[3] Mats Bengtsson and Björn Ottersten, "Optimal downlink beamforming using semidefinite optimization," in *Proc. 37th Annual Allerton Conference on Communication, Control, and Computing*, Sept. 1999, pp. 987–996.

[4] David Samuelsson, Mats Bengtsson, and Björn Ottersten, "Optimal downlink beamforming with additional constraints," in *Proc.* 37^{th} *Asilomar Conf. on Signals, Systems, and Computers*, Nov 2003, vol. 2, pp. 1811 – 1814.

[5] Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.