COMPARISON OF THE BHATTACHARYYA AND CRAMER-RAO LOWER BOUNDS FOR THE POSITION ESTIMATION OF AN OFDM TRANSMITTER

Marilynn P. Wylie-Green

Nokia Research Center
6000 Connection Drive
Irving, TX 75039

ABSTRACT

In this paper, we consider a comparison of the Bhattacharyya Bound and the Cramer-Rao Lower Bound for the problem of mobile positioning. Since the Cramer-Rao Lower Bound is known to be optimistic, we investigate the improvement that may be obtained by computing the third and fifth order Bhattacharyya Bounds as alternate benchmarks for performance. The analytical results derived in this paper are then applied to the problem of locating an Ultrawideband OFDM transmitter in a line of sight environment. Our results show that the third order and fifth order Bhattacharyya bounds respectively offer a 1.9 dB and a 2.3 dB improvement over the CRLB over the range of SNRs that have been deemed feasible for the Ultrawideband system.

1. INTRODUCTION

The problem of mobile location estimation is becoming increasingly important for the delivery of new location-based services and applications. The Cramer-Rao Lower Bound [1] (CRLB), which is typically used as a benchmark for mean squared error performance, has often been used to facilitate comparisons between different positioning estimators. However, it is known to be an optimistic bound that may not be tight enough in certain SNR regions to provide meaningful insight into the achievable estimator performance [2 – 4]. Hence, there are other lower bounds that may, for a certain region of consideration, be better indicators of estimator performance, and for that reason this paper investigates the usefulness of the Bhattacharyya Bound (BB) for position estimation.

The main contribution of this paper is in the presentation of the third and fifth order BB for position estimation error, with special application to Ultrawideband OFDM transmissions. Although the BB has been calculated for the problem of time delay estimation [2], this paper represents (to the best knowledge of the author) a new contribution in the formal derivation of this bound for the location estimation problem. The objective of our study is to investigate the tightness of the BB vis-à-vis the CRLB as a function of feasible SNRs for an Ultrawideband OFDM system.

The rest of this paper is organized as follows. In Section 2, we present the OFDM signal model as well as the model for signal reception. Sections 3 and 4 respectively discuss the CRLB and the BB. In Section 5, we derive the BB and CRLB for position estimation for and OFDM signal. Finally, in Section 6 we present a conclusion of this study.

2. SIGNAL MODEL

We consider a wireless device located at (x, y) that transmits a single OFDM symbol which is received by M synchronized receivers that are located at (x_m, y_m), for m = 1…M. The complex envelope of an OFDM symbol is generally modeled as:

$$s(t) = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} C_k e^{j\omega_k t} \qquad 0 \le t \le T \tag{1}$$

where K represents the number of OFDM sub-carriers, T denotes the OFDM symbol interval and $\omega_k = 2\pi(k-K/2)/T$ is the (relative) sub-carrier frequency. The complex coefficients, C_k, may correspond to data, training or pilot symbols. The signal that is measured at the m-th receiving device (m = 1…M) is modeled as:

$$r_m(t) = A_m s(t-\tau_m) + n_m(t) \qquad 0 \le t \le T_1 \tag{2}$$

where A_m generally denotes a complex signal amplitude and $n_m(t)$ is complex Gaussian noise with spectral density N_0. T_1 is the observation time, where $T_1 >> T$. The propagation delay, τ_m is defined as:

$$\tau_m = \frac{1}{c}\sqrt{(x-x_m)^2 + (y-y_m)^2} \tag{3}$$

where c = 3 × 10^8 m/s is the speed of light. The location system can use the measured propagation delays and the known geometry of receiving devices in order to estimate the position of the OFDM transmitter.

3. CRAMER-RAO LOWER BOUND

The Cramer-Rao Lower Bound provides a lower bound on the covariance of any unbiased estimator, and has become one of the most popular methods for performance comparison. Let $\hat{\boldsymbol{\theta}}(\mathbf{r})$ be an estimate of the unknown, non-random parameters in the N × 1 vector $\boldsymbol{\theta}$ and denote $E_{\boldsymbol{\theta}}\{\cdot\}$ as the expected value conditioned on $\boldsymbol{\theta}$. The vector $\mathbf{r}$ represents the measurement vector. Then, the CRLB can be expressed as [1]:

$$E_{\boldsymbol{\theta}}\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{r})-\boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{r})-\boldsymbol{\theta}\right)^T\right\} \geq \mathbf{I}^{-1}, \tag{4}$$

where $\mathbf{I}$ is the N × N Fisher's Information Matrix (FIM) whose (*m, n*)-th element is defined as:

$$\mathbf{I}_{m,n} = E_{\boldsymbol{\theta}}\left\{\frac{\partial \ln f_{\boldsymbol{\theta}}(\mathbf{r})}{\partial \theta_m}\cdot\frac{\partial \ln f_{\boldsymbol{\theta}}(\mathbf{r})}{\partial \theta_n}\right\}, \quad \text{for } m,n = 1,2,\cdots,N, \tag{5}$$

and $f_\theta(\mathbf{r})$ is the joint probability density function of the measured data conditioned on the unknown parameter vector $\boldsymbol{\theta}$. For the problem under consideration in which the measurement noise is complex Gaussian with zero mean and spectral density N_0, it is given by:

$$f_{\boldsymbol{\theta}}(\mathbf{r}) \propto \prod_{m=1}^{M} \exp\left\{-\frac{1}{N_0}\int\left|r_m(t)-A_m s(t-\tau_m)\right|^2 dt\right\}. \tag{6}$$

In the next section, we present the Bhattacharyya bound for parameter estimation.

4. BHATTACHARYYA BOUND

Let $\hat{\boldsymbol{\theta}}(\mathbf{r})$ be an estimate of the vector of the unknown parameters in the N × 1 vector, $\boldsymbol{\theta}$. Then, the BB can be expressed as [1]:

$$E_{\boldsymbol{\theta}}\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{r})-\boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{r})-\boldsymbol{\theta}\right)^T\right\} \geq \mathbf{J}_P^{(1,1)}. \tag{7}$$

$\mathbf{J}_P^{(1,1)}$ is the N x N matrix that is found in the upper left corner of the matrix inverse, $\mathbf{J}_P^{-1}$, where $\mathbf{J}_P$ is the NP x NP matrix:

$$\mathbf{J}_P = \begin{bmatrix} \mathbf{I}_{1,1} & \cdots & \mathbf{I}_{1,P} \\ \vdots & \ddots & \vdots \\ \mathbf{I}_{P,1} & \cdots & \mathbf{I}_{P,P} \end{bmatrix}. \tag{8}$$

The $(m, n)^{th}$ element of the N x N matrix $\mathbf{I}_{k,r}$ is defined as:

$$\left[\mathbf{I}_{k,r}\right]_{m,n} = E_{\boldsymbol{\theta}}\left\{\frac{\partial^k \ln f_{\boldsymbol{\theta}}(\mathbf{r})}{\partial \theta_m^k}\cdot\frac{\partial^r \ln f_{\boldsymbol{\theta}}(\mathbf{r})}{\partial \theta_n^r}\right\} \tag{9}$$

for k, r = 1…P and m, n = 1…N. It is interesting to note that $\mathbf{I}_{1,1}$ is equal to the FIM, so that the BB reduces to the CRLB for P = 1. In the following section, we apply the BB to the problem of mobile location estimation and show that it does provide a tighter lower bound than the CRLB.

5. LOWER BOUNDS FOR POSITIONING

In this section, we present the CRLB and BB for the problem of position estimation when a wireless device transmits a single OFDM symbol to M synchronized receiving devices. The extension of these results to the case in which there is more than one transmitted symbol is straightforward.

We consider estimation of the non-random parameter vector of transmitter coordinates, $\boldsymbol{\theta} = [x \quad y]^T$. An application of the chain rule for differentiation allows us to calculate $\mathbf{J}_P$ in the following manner:

$$\mathbf{J} = \begin{bmatrix} \mathbf{I}_{1,1} & \cdots & \mathbf{I}_{1,P} \\ \vdots & \ddots & \vdots \\ \mathbf{I}_{P,1} & \cdots & \mathbf{I}_{P,1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\boldsymbol{\Psi}_{1,1}\mathbf{A}_1^T & \cdots & \mathbf{A}_1\boldsymbol{\Psi}_{1,P}\mathbf{A}_P^T \\ \vdots & \ddots & \vdots \\ \mathbf{A}_P\boldsymbol{\Psi}_{P,1}\mathbf{A}_1^T & \cdots & \mathbf{A}_P\boldsymbol{\Psi}_{P,P}\mathbf{A}_P^T \end{bmatrix}, \tag{10}$$

where the M x M matrix, $\boldsymbol{\Psi}_{m,n} = E_{\boldsymbol{\theta}}\left\{\mathbf{u}_m\mathbf{u}_n^T\right\}$. The elements of the 2 x M matrix $\mathbf{A}_n$ and the M x 1 vector u_n are respectively defined as:

$$\begin{aligned} \left[\mathbf{A}_n\right]_{k,r} &= \frac{\partial^n \tau_r}{\partial \theta_k^n} & n = 1,\cdots,P;\ k = 1,2;\ r = 1,\cdots,M \\ \left[\mathbf{u}_n\right]_{r,1} &= \frac{\partial^n \ln f_{\boldsymbol{\theta}}(\mathbf{r})}{\partial \tau_r^n} & n = 1,\cdots,P;\ r = 1,\cdots,M \end{aligned} \tag{11}$$

For this study, we have derived analytical expressions for calculation of the BB for P ≤ 5. Let us define $R_k = \sqrt{(x-x_k)^2 + (y-y_k)^2}$. Then, based on the definitions in (3) and (11) we can derive the explicit expressions for the elements of $\mathbf{A}_n$ as follows:

$$
\begin{aligned}
[\mathbf{A}_1]_{1,k} &= \frac{x - x_k}{cR_k} \\
[\mathbf{A}_2]_{1,k} &= -\frac{(x - x_k)^2}{cR_k^3} + \frac{1}{cR_k} \\
[\mathbf{A}_3]_{1,k} &= \frac{3(x - x_k)^3}{cR_k^5} - \frac{3(x - x_k)}{cR_k^3} \\
[\mathbf{A}_4]_{1,k} &= -\frac{3}{cR_k^3} + \frac{18(x - x_k)^2}{cR_k^5} - \frac{15(x - x_k)^4}{cR_k^7} \\
[\mathbf{A}_5]_{1,k} &= \frac{45(x - x_k)}{cR_k^5} - \frac{150(x - x_k)^3}{cR_k^7} + \frac{105(x - x_k)^5}{cR_k^9}
\end{aligned}
\tag{12}
$$

The second row of elements, i.e. $[\mathbf{A}_n]_{2,k}$, can be found by substituting y for x and y_k for x_k in (12). The elements of the vectors, $\mathbf{u}_n$, may be calculated by consideration of the definitions found in (6) and (11) permits us to formulate the expressions for $\mathbf{u}_n$ as follows:

$$
\begin{aligned}
[\mathbf{u}_1]_{m,1} &= -\frac{2}{N_0}\mathrm{Re}\left\{ \int A_m \dot{s}(t-\tau_m)\left[r_m^*(t) - A_m^* s^*(t-\tau_m)\right]dt \right\} \\
[\mathbf{u}_2]_{m,1} &= \frac{2}{N_0}\mathrm{Re}\left\{ \int A_m \ddot{s}(t-\tau_m)\left[r_m^*(t) - A_m^* s^*(t-\tau_m)\right]dt \right\} \\
&- \frac{2}{N_0}|A_m|^2 \int |\dot{s}(t-\tau_m)|^2 dt \\
[\mathbf{u}_3]_{m,1} &= \frac{-2}{N_0}\mathrm{Re}\left\{ \int A_m \dddot{s}(t-\tau_m)\left[r_m^*(t) - A_m^* s^*(t-\tau_m)\right]dt \right\} \\
&+ \frac{6}{N_0}|A_m|^2\,\mathrm{Re}\left\{ \int \dot{s}^*(t-\tau_m)\ddot{s}(t-\tau_m)dt \right\} \\
[\mathbf{u}_4]_{m,1} &= \frac{2}{N_0}\mathrm{Re}\left\{ \int A_m s^{(4)}(t-\tau_m)\left[r_m^*(t) - A_m^* s^*(t-\tau_m)\right]dt \right\} \\
&- \frac{8}{N_0}\mathrm{Re}\left\{ \int | A_m |^2\, \dddot{s}(t-\tau_m)\dot{s}^*(t-\tau_m)dt \right\} \\
&- \frac{6}{N_0}\int | A_m |^2\, |\ddot{s}(t-\tau_m)|^2 dt \\
[\mathbf{u}_5]_{m,1} &= -\frac{2}{N_0}\mathrm{Re}\left\{ \int A_m s^{(5)}(t-\tau_m)\left[r_m^*(t) - A_m^* s^*(t-\tau_m)\right]dt \right\} \\
&+ \frac{6}{N_0}\mathrm{Re}\left\{ \int | A_m |^2\, s^{(4)}(t-\tau_m)\dot{s}^*(t-\tau_m)dt \right\} \\
&+ \frac{8}{N_0}\mathrm{Re}\left\{ \int | A_m |^2\, \dddot{s}(t-\tau_m)\ddot{s}^*(t-\tau_m)dt \right\}
\end{aligned}
\tag{13}
$$

For the OFDM signal under consideration, we can express the cross-energy terms as:

$$
\mathrm{Re}\left\{ \int_T s^{(m)}(t)\left(s^{(n)}(t)\right)^* dt \right\} = \mathrm{Re}\left\{ \frac{j^{m+n}(-1)^n T}{K} \sum_{k=0}^{K-1} |C_k|^2 \omega_k^{m+n} \right\}
$$

$$
= \begin{cases} 0 & m+n \text{ odd} \\ \dfrac{\mathrm{Re}\{j^{m+n}\}(-1)^n T}{K} \displaystyle\sum_{k=0}^{K=1} \omega_k^{m+n} & m+n \text{ even} \end{cases}
\tag{14}
$$

For constant modulus modulation schemes, $|C_k|^2=1$, and the elements of the symmetrical matrix $[\mathbf{\Psi}_{m,n}]$ simplify to:

$$
\begin{aligned}
[\mathbf{\Psi}_{1,1}]_{m,n} &= \frac{2|A_m|^2 T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^2 \delta_{mn} \\
[\mathbf{\Psi}_{1,3}]_{m,n} &= -\frac{2|A_m|^2 T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^4 \delta_{mn} \\
[\mathbf{\Psi}_{1,5}]_{m,n} &= \frac{2|A_m|^2 T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^6 \delta_{mn} \\
[\mathbf{\Psi}_{2,2}]_{m,n} &= \frac{2|A_m|^2 T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^4 \delta_{mn} + \frac{4|A_m|^2|A_n|^2 T^2}{N_0^2 K^2}\left(\sum_{k=0}^{K-1}\omega_k^2\right)^2 \\
[\mathbf{\Psi}_{2,4}]_{m,n} &= -\frac{2\,|\,A_m\,|^2\,T}{N_0 K}\sum_{k=0}^{K=1}\omega_k^6 \delta_{mn} - \frac{4\,|\,A_m\,|^2|\,A_n\,|^2\,T^2}{N_0^2 K^2}\sum_{k=0}^{K=1}\omega_k^2 \sum_{k=0}^{K=1}\omega_k^4 \\
[\mathbf{\Psi}_{3,3}]_{m,n} &= \frac{2|A_m|^2 T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^6 \delta_{mn} \\
[\mathbf{\Psi}_{3,5}]_{m,n} &= -\frac{2\,|\,A_m\,|^2\,T}{N_0 K}\sum_{k=0}^{K=1}\omega_k^8 \delta_{mn} \\
[\mathbf{\Psi}_{4,4}]_{m,n} &= \frac{2\,|\,A_m\,|^2\,T}{N_0 K}\sum_{k=0}^{K=1}\omega_k^8 \delta_{mn} + \frac{4\,|\,A_m\,|^2|\,A_n\,|^2\,T^2}{N_0^2 K^2}\left(\sum_{k=0}^{K=1}\omega_k^4\right)^2 \\
[\mathbf{\Psi}_{5,5}]_{m,n} &= \frac{2\,|\,A_m\,|^2\,T}{N_0 K}\sum_{k=0}^{K-1}\omega_k^{10} \delta_{mn}
\end{aligned}
\tag{15}
$$

where $\delta_{mn} = 1$ when m = n and is zero otherwise. We note that $[\mathbf{\psi}_{k,j}]_{m,n} = 0$ for j+k odd. By appropriately partitioning the matrix $\mathbf{J}_P$, we can show that $\mathbf{J}_P^{(1,1)}$ is the 2 x 2 matrix that is calculated as:

$$
\mathbf{J}_P^{(1,1)} = \left(\mathbf{A}_1\mathbf{\Psi}_{11}\mathbf{A}_1^T - \mathbf{B}_P\mathbf{Y}_P^{-1}\mathbf{B}_P^T\right)^{-1}
\tag{16}
$$

where

$$
\begin{aligned}
\mathbf{B}_P &= \left[\mathbf{A}_1\mathbf{\Psi}_{12}\mathbf{A}_2^T \quad \cdots \quad \mathbf{A}_1\mathbf{\Psi}_{1P}\mathbf{A}_P^T\right] \\
\mathbf{Y}_P &= \begin{bmatrix} \mathbf{A}_2\mathbf{\Psi}_{22}\mathbf{A}_2^T & \cdots & \mathbf{A}_2\mathbf{\Psi}_{2P}\mathbf{A}_P^T \\ \vdots & \ddots & \vdots \\ \mathbf{A}_P\mathbf{\Psi}_{2P}\mathbf{A}_2^T & \cdots & \mathbf{A}_P\mathbf{\Psi}_{PP}\mathbf{A}_P^T \end{bmatrix}
\end{aligned}
\tag{17}
$$

By comparison, the CRLB is calculated as:

$$
\mathbf{I}^{-1} = \left(\mathbf{A}_1\mathbf{\Psi}_{11}\mathbf{A}_1^T\right)^{-1}
\tag{18}
$$

The second, nonnegative term, in (16) therefore results in the improvement of the BB vis-à-vis the CRLB. For the purposes of numerical analysis, we can compute the final positioning error as the square root of the sum of the two diagonal terms in $\mathbf{J}^{(1,1)}$ and $\mathbf{I}^{-1}$.

6. NUMERICAL ANALYSIS

In this study, the OFDM parameter values are extracted from [5], which provides a link budget analysis for the MB-OFDM approach that has been proposed to the IEEE 802.15.3a body for use in WPANs. We assume K = 128, T = 242.42 ns, an average transmit power level of –10.3 dBm and a reference path loss at 1 meter of 44.2 dB. Beyond 1 m, the path loss L = $20\log_{10}d$. The implementation loss is I = 2.5 dB and the noise figure is 6.6 dB. N_0, in dBm/Hz, is -174. The noise power per bit is given by $P_N = -174$ dBm/Hz + $10\log_{10}(R_b)$ + NF, where R_b denotes the data rate and NF = 6.6 dB is the noise figure. R_b = 110 Mbps and 480 Mbps, for which 10m and 2m are, respectively, the maximum distances of operation. x = y = 0 and the M = 4 receivers are all located on a circle surrounding it. P_R is received power.

Fig. 1 – 2 all compare the CRLB to BB_3 (P = 3) and BB_5 (P = 5) for the two data rates investigated. The range of SNRs considered in each figure corresponds to the minimum separation of 1 m to the maximum separation distance that is suggested for each data rate. In all of these cases, BB_5 only provides a modest improvement over BB_5, but BB_3 provides a 1.9 dB improvement over the CRLB and BB5 provides a 2.3 dB improvement over the CRLB. We note that for the range of SNRs over which this system is intended to operate, that there is no convergence of these performance bounds. We also observe that the increase in data rate brings improvements in the mean squared error, as is evident from a direct comparison of Fig. 1 – 2. We note that our calculations are relevant for one observed OFDM symbol interval. One can easily show that the performance bound decreases proportionally to the number of observed symbol intervals.

7. CONCLUSION

In this paper, we have presented the BB for position estimation of an OFDM signal in a line-of-sight environment and compared it to the CRLB. The parameters selected for this study coincide with those that are used in the MB-OFDM approach (over one OFDM symbol interval). The numerical results, which are based on calculation of the third order and fifth order BB indicates that over the region of SNRs for which this Ultrawideband system will operate, that BB_3 offers a 1.8 dB improvement and BB_5 offers a 2.3 dB improvement on the mean squared error when compared to the CRLB.

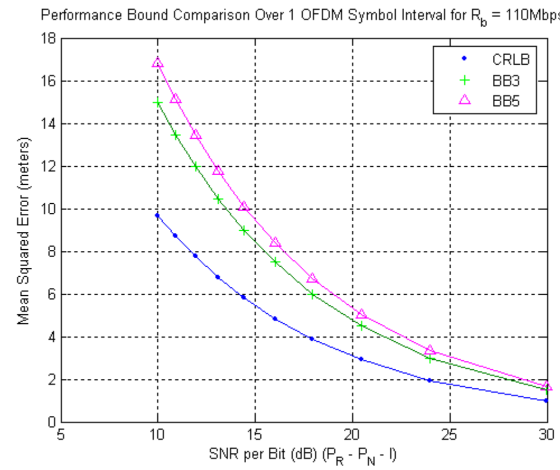

Figure 1.Performance bounds for R_b=110 Mbps.

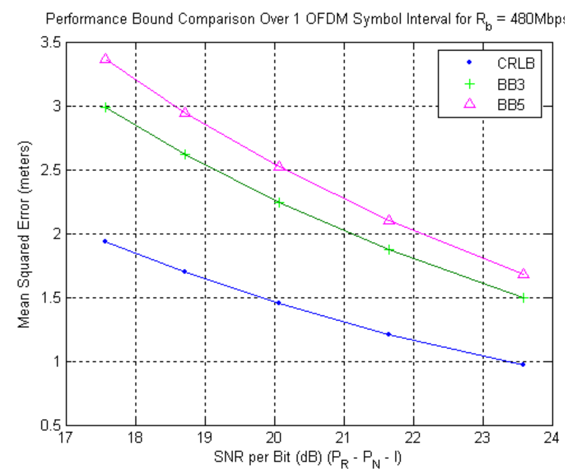

Figure 2.Performance bounds for R_b=480 Mbps.

8. REFERENCES

[1] H. L. Van Trees, *Detection, Estimation and Modulation Theory*. New York: Wiley New, 1968.

[2] F. Lu and J. Krogmeier, "Modified Bhattacharyya Bounds and Their Application to Timing Estimation", *Wireless Communications and Networking Conference*, WCNC 2002, pp. 242 – 248, March 2002.

[3] A. Bhattacharyya, "On some analogues of the amount of information and their use in statistical estimation," *SANKHYA,* vol. 8, pp. 1946 – 1948.

[4] R. Miller and C. B. Chang, "A Modified Cramer-Rao bound and its application," *IEEE Transactions on Information theory*, vol. IT-24, pp. 398 – 400, May 1978.

[5] A. Batra, J. Balakrishnan, R. Aiello, J. Foerster and A. Dabak, "Design of a Multiband OFDM System for Realistic UWB Channel Environments", *IEEE Transactions on Microwave Theory and Techniques*, vol. 52, No. 9, pp. 2123 – 2138, Sept. 2004.