FREQUENCY ESTIMATION FROM MULTIPLE LAGS OF CORRELATIONS IN THE PRESENCE OF MA COLORED NOISE

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ABSTRACT

This paper provides a new frequency estimator based on the multiple lags of autocorrelations with discrete Fourier transform (DFT) phase unwrapping. It is proved that this estimator is efficient for short data length while maintaining a low signal noise ratio (SNR) threshold. It is also shown that this estimator is statistically similar to maximum likelihood estimator (MLE), but with lower computational load. Furthermore, the estimator can also be applied to frequency estimation in the presence of moving average (MA) colored noise. In this case, it is shown that the variance of the estimator achieves the Cramér-Rao bound (CRB) asymptotically.

1. INTRODUCTION

Estimating the frequency of a single sinusoid in additive white Gaussian noise (AWGN) arises in many signal applications [1]. MLE is known to be an excellent estimator especially at low SNR. However, the computation burden involved in the search of the maximum is quite large and often sub-optimal methods are used. The pulse pair method is a simple technique which is widely used in radar and sonar applications [2–5], but it suffers from a high SNR threshold. (In this paper we use the term "SNR threshold" to identify the SNR value where estimation accuracy starts to deviate appreciably from the CRB). As an extension to the pulse pair method, frequency estimation from proper sets of correlations is investigated in [6]. By choosing a proper set of correlations, the accuracy of the estimator was improved. One disadvantage of using multiple lag correlations is the introduction of phase unwrapping. To resolve the phase ambiguity, the Chinese remainder theorem (CRT) was used in [6, 7], but the SNR threshold is still higher than MLE. In this paper, we propose a new technique based on the use of multiple lags of correlations in conjunction with DFT. This estimator is proposed as an efficient frequency estimator for short data samples and its SNR threshold value is comparable to that of MLE. More importantly, it is shown that by using the correlations of higher lag values, this estimator can also be applied to the efficient frequency estimation in the presence of MA colored noise.

Estimation of parameters in the autoregressive(AR) noise was discussed in [8]. It was shown that MLE estimation of the parameters in the noise signal model is approximately efficient. But it leads to an involved computational problem, thus the nonlinear least squares (NLS) technique was proposed as a simple frequency estimator. Though it doesn't require to estimate the noise parameters, the data samples need to be zeropadded which affects the accuracy of the NLS estimator [9]. In this paper, we propose a novel estimator which is unbiased in the presence of MA colored noise. It is proven that the variance of the estimator achieves CRB asymptotically. The SNR threshold of the estimator is comparable to MLE but it avoids the fine grid search of the MLE, hence is computationally efficient.

2. FREQUENCY ESTIMATION FROM MULTIPLE LAGS OF CORRELATIONS

Consider the signal received in AWGN channel

$$x(n) = Ae^{j\phi}e^{j\omega_0 n} + v(n), \quad n = 1, 2, ...N - 1. \tag{1}$$

where $v(n)$ is a complex circular white Gaussian noise process with variance σ^2, ϕ is uniformly distributed in $(0, 2\pi]$. ω_0 is the angular frequency and $\omega_0 \in [-\pi, \pi)$. A is the signal amplitude and N is the number of data samples. SNR is defined as A^2/σ^2. The angular frequency can be estimated from the mth lag autocorrelation as

$$\hat{\omega}_m + 2\pi l = \frac{1}{m}\arg\{\widehat{r}(m)\}, \tag{2}$$

for some integer l due to the nature of phase unwrapping, where $\arg\{\cdot\}$ represents the argument of the complex number, $\widehat{r}(m)$ is the autocorrelation of the received signal given by

$$\widehat{r}(m) = \frac{1}{N-m}\sum_{n=1}^{N-m} x(n+m)x^*(n) \tag{3}$$

It is known that the estimator in (2) is unbiased and its variance achieves a minimum value for $m = 2N/3$ which is 0.51dB above CRB [3, 4]. We note that by considering the multiple lags of correlations, the accuracy of the estimation can be improved. With proper phase unwrapping, the frequency can be estimated unbiasedly from every correlation lag m. Assume that it is $\hat{\omega}_m, m = 1, 2..K, K \leq N - 1$. Then a weighted linear estimator can be defined as in [6]

$$\widehat{\omega}_0 = \sum_{m=1}^{K} a_m\hat{\omega}_m, \quad m = 1, 2....K, K \leq N - 1. \tag{4}$$

where a_m's are weight coefficients yet to be determined. For simplicity, we restricts the estimator to be linear in the estimated frequency data set. The variance of the estimator in (4) can be expressed as

$$E[(\omega_0 - \widehat{\omega}_0)^2] = \sum_{m=1}^{K-1}\sum_{l=1}^{K-1}\frac{a_m a_l E[Y_m Y_l]}{m(N-m)l(N-l)} \tag{5}$$

where $E[Y_m Y_l] = \frac{\sigma^2 \min(N-m,m,N-l,l)}{A^2}$ [10].

2.1. Optimal Window Determination and Performance Analysis

The optimal weighting window for the linear estimator can be obtained as in [1]

$$\mathbf{a} = \frac{\mathbf{R}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{R}^{-1}\mathbf{1}} \tag{6}$$

where $\mathbf{a} = [a_1, a_2, ...a_K]^T$, $\mathbf{1} = [1, 1, ...1]^T$ and $\mathbf{R}$ is the covariance matrix of the frequency estimators from different correlation lags. The (m, l)th element of $\mathbf{R}$ was derived in [10] as

$$\mathbf{R}_{m,l} = \frac{\min(K-m, K-l, m, l)}{(K-m)m(K-l)l\mathrm{SNR}}. \tag{7}$$

It was noted in [6] that $\mathbf{R}$ is rank deficient for $K > N/2$ and it is not relevant to consider the case when $K > N/2$ for complex reasons. But here we show that due to the symmetrical property of $\mathbf{R}_{m,l}$, we may first obtain half of the coefficients $a_m, m = 1, 2, ...(N-1)/2$ (without loss of generality, assume N is odd) by inverting $\mathbf{R}_{m,l}(m, l \leq (N-1)/2)$. Then their mirror coefficients are obtained symmetrically around $(N-1)/2$. By a polynomial curve fitting technique given in [1], a_m can be obtained as a parabolic window

$$a_m = \frac{6m(N-m)}{N(N^2-1)} \qquad m = 1, ...N-1. \tag{8}$$

Considering the symmetry property of the window and covariance matrix, we will only use $(N-1)/2$ number of correlation lags to estimate the frequency. Thus (5) for $K = (N-1)/2$ becomes

$$E(\varepsilon^2) = \frac{144}{\mathrm{SNR}N^2(N^2-1)^2} \sum_{m=1}^{\frac{N-1}{2}} \sum_{l=1}^{\frac{N-1}{2}} \min(m, l) \tag{9}$$

which leads to

$$E(\varepsilon^2) = \frac{6}{\mathrm{SNR}N(N^2-1)} \tag{10}$$

This is the CRB in the presence of AWGN. The window to achieve CRB is expressed as

$$a_m = \frac{12m(N-m)}{N(N^2-1)} \qquad m = 1, 2, ...\frac{N-1}{2}. \tag{11}$$

Using a similar approach, for arbitrary K, (11) can be expressed as

$$a_m = \frac{3m(2K+1-m)}{K(K+1)(2K+1)}, \;\; m = 1, 2...K, K \leq \frac{N-1}{2} \tag{12}$$

which is the same window derived in [6] using complicated recursions. Furthermore, we can easily prove that the estimator with the following mirror coefficients of (11) will also achieve CRB

$$a_m = \frac{12m(N-m)}{N(N^2-1)} \quad m = \frac{N+1}{2}, \frac{N+3}{2}, ...N-1. \tag{13}$$

Our novel estimator proposed in this paper is based on the mirror coefficients. Simulation also confirms that the estimator with (13) achieves CRB. At this point, we note that estimator in (4), and thus both estimators using (11) and (13) need proper phase unwrapping. However, in the presence of AWGN when CRT is used to resolve phase unwrapping, the SNR threshold of the estimator obtained from (13) will be much higher than that of (11). In this paper, we shall use the DFT as the initial coarse search to resolve phase unwrapping due to the excellent threshold characteristics of DFT [11], so that (11) and (13) will lead to the same SNR threshold. More importantly, we shall show in the next Section that (13) can also be used efficiently in the frequency estimation in MA colored noise. Fig.1 is the simulation of the mean square error (MSE) of the estimator using multiple correlations ($K = 1, 2...(N-1)/2$) for different phase unwrapping techniques. It is shown that the estimator with DFT phase unwrapping is efficient and have lower SNR threshold compared to CRT in [6] and Crozies phase unwrapping methods in [12]. From Fig.1, we conclude that the MSE results using the coefficients (11) with DFT phase unwrapping is as same as MLE, and use of the coefficients in (13) results in identical performance.

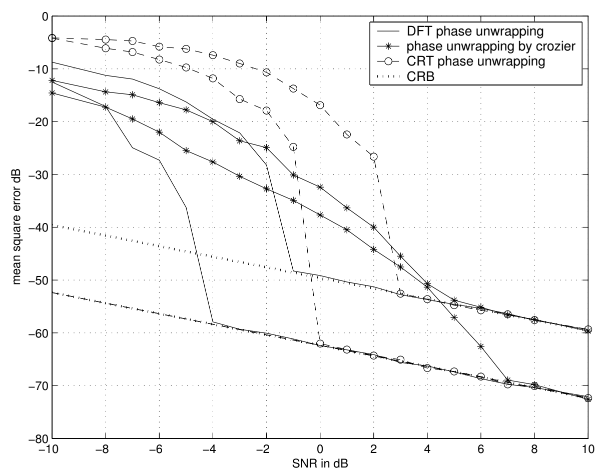

Fig. 1. The Performance of frequency estimation as SNR varies, for DFT phase unwrapping (solid line), CRT phase unwrapping (circled line) and Crozier's phase unwrapping (stared line). $N = 24, 64$, Normalized frequency $f = 0.124$, compared with CRB(dashed line)

2.2. Relation With MLE

The optimal solution of the MLE frequency estimate results from locating the peak of the periodogam. That is

$$\widehat{\omega}_0 = \max\{I(\omega)\} \tag{14}$$

where $I(\omega)$ is expressed as

$$I(\omega) = |\sum_{n=0}^{N-1} x(n)e^{-j\omega n}|^2 = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} x(n)x^*(k)e^{-j\omega(n-k)} \tag{15}$$

To find the maximum of the periodogram, take the derivative of (15) with respect to ω and set it to zero. Let $n - k = m$ and after some manipulation, we obtain

$$\sum_{m=0}^{N-1} m(N-m)\mathrm{Im}\{\widehat{r}(m)e^{-j\widehat{\omega}_0 m}\} = 0 \qquad (16)$$

where $\mathrm{Im}\{\cdot\}$ is the imaginary part of a complex number, $\widehat{r}(m)$ is defined in (3). It can be shown that

$$\mathrm{Im}\{\widehat{r}(m)e^{-j\hat{\omega}_0 m}\} \approx m\hat{\omega}_0 - \arg\{\widehat{r}(m)\} \qquad (17)$$

Insert (17) into (16), the weighted autocorrelation based frequency estimate is obtained as

$$\hat{\omega}_0 \approx \frac{\sum_{m=0}^{N-1}(N-m)m\arg\{\widehat{r}(m)\}}{\sum_{m=0}^{N-1}(N-m)m^2} \qquad (18)$$

Thus the window related to MLE is given by

$$a_m = \frac{(N-m)m^2}{\sum_{m=0}^{N-1}(N-m)m^2} \qquad (19)$$

The MLE relation by examining the periodogram was also discussed in [13]. But the window obtained in [13] is not optimal. We notice that (19) approximately is a parabolic window, especially for large N. We have proved that the parabolic window in (11) is optimal and the variance of the estimator is equal to CRB. As such, it is seen that the estimator with (19) by examining the periodogram is approximately efficient, while the estimator proposed with the optimal window in (11) is efficient for any length of data samples.

3. FREQUENCY ESTIMATION IN THE PRESENCE OF MA COLORED NOISE

Assume the noise term in the signal model (1) is colored and can be modelled as an order q MA process, which is expressed as

$$e(n) = \sum_{k=1}^{q} b_k w(n-k) + w(n) \qquad (20)$$

where $w(n)$ is a white noise process with variance σ^2. $e(n)$ is obtained by passing AWGN through a filter with system function $B(z) = 1 + b_1 z^{-1} + ... b_q z^{-q}$. It is usually assumed that the filter $B(z)$ is minimum phased and the zeros are within the unit circle. Then the spectral density of $e(n)$ is given by

$$\psi(\omega) = |1 + \sum_{k=1}^{q} b_k e^{j\omega k}|^2 \qquad (21)$$

Before deriving the variance of the frequency estimator from MA colored noise, define: $\mathbf{b} = [1 \quad b_1 e^{-j\omega_0} .....b_q e^{-jq\omega_0}]^T$, $\mathbf{1} = [1 \quad 1.....1]^T$, $\mathbf{w} = [w(n) \quad w(n-1)....w(n-q)]^T$, where $[\cdot]^T$ denoting transposition. Then the signal model can be expressed as.

$$x(n) = (Ae^{j\phi} + v(n))e^{j\omega_0 n} \qquad (22)$$

where $v(n) = \mathbf{b}^T\mathbf{w}$ and $v(n)e^{j(\omega_0 n + \phi)} = e(n)$, so that $v(n)$ has the same distribution with $e(n)$. Then the estimated autocorrelation is expressed as

$$r(m) = \frac{A^2}{N-m}\sum_{n=1}^{N-m}(1 + \frac{v(n+m)+v^*(n)}{A} + \frac{v^*(n)v(n+m)}{A^2})e^{j\omega_0 m} \qquad (23)$$

The error of the estimator from the mth lag correlation is given by

$$\varepsilon_m = \frac{1}{m}\arg\{1 + \frac{X_m}{N-m} + j\frac{Y_m}{N-m}\} \qquad (24)$$

where $X_m = \sum_{n=1}^{N-m} t_1(n+m) + t_1(n) + t_1(n)t_1(n+m) + t_2(n)t_2(n+m)$, $Y_m = \sum_{n=1}^{N-m} t_2(n+m) - t_2(n) - t_1(n)t_2(n+m) + t_2(n)t_1(n+m)$. $t_n = v(n)/A^2$, $t(n) = t_1(n) + jt_2(n)$. Since $t_1(n)$ and $t_2(n)$ are both MA colored process, $E(\varepsilon_m) \neq 0$ when $m \leq q$. It can be easily proven that when $m > q$, $E(X_m) = E(Y_m) = 0$. Thus

$$\varepsilon_m \approx \frac{Y_m}{m(N-m)} \quad \text{for} \quad m > q \qquad (25)$$

Now we shall calculate the variance of the estimator in MA colored noise. The covariance function of Y_m is required to proceed with the performance evaluation. Note that $t_1(n) = \mathbf{b}^T\mathbf{w}/2A^2$ and denote $t_1(n+m) = \mathbf{b}^T\mathbf{w}z^m/2A^2$. Through further manipulation, we can obtain

$$E[Y_m Y_l] = \frac{\mathbf{b}^T}{A^2}\left(\sum_{n_1=1}^{N-m}\sum_{n_2=1}^{N-l} E((\mathbf{w}_1 z^m)(\mathbf{w_2} z^l)^T - (\mathbf{w}_1 z^m)\mathbf{w_2}^T)\right)\mathbf{b} \qquad (26)$$

where $E[(\mathbf{w}_1 z^m)(\mathbf{w_2} z^l)^T - (\mathbf{w}_1 z^m)\mathbf{w_2}^T]$ is a $q \times q$ matrix, which can be expressed as

$$E((\mathbf{w}_1 z^m)(\mathbf{w_2} z^l)^T - (\mathbf{w}_1 z^m)\mathbf{w_2}^T) = \mathbf{D}_1 - \mathbf{D}_2 \qquad (27)$$

with

$$[\mathbf{D}_1]_{i,j} = \delta_{n_1+m-i,n_2+l-j}, \ i,j = 1,2,...q. \qquad (28)$$

$$[\mathbf{D}_2]_{i,j} = \delta_{n_1+m-i,n_2-j}, \ i,j = 1,2,...q. \qquad (29)$$

Define $\mathbf{M} = \sum_{n_1=1}^{N-m}\sum_{n_2=1}^{N-l}(\mathbf{D}_1 - \mathbf{D}_2)$ and after further derivation, $\mathbf{M}$ can be expressed as

$$[\mathbf{M}]_{ij} = \begin{cases} \min(N-m, m, N-l, l), & i = j \\ \min(N-m+t, m-t, N-l, l), & |i-j| = t, \end{cases} \qquad (30)$$

where $t = 1, 2...q-1$. Thus we can get

$$E[Y_m Y_l] = \frac{1}{A^2}\mathbf{b}^T\mathbf{M}\mathbf{b} \qquad (31)$$

Using the optimal window as given in (13) for unbiased frequency estimation, we can obtain the corresponding variance as following

$$E(\varepsilon^2) = \frac{144}{A^2N^2(N^2-1)^2}\sum_{m=(N+1)/2}^{N-1}\sum_{l=(N+1)/2}^{N-1}\mathbf{b}^T\mathbf{M}\mathbf{b} \qquad (32)$$

where $\mathbf{M}$ is the $q \times q$ square matrix and defined in (30). Assume $N >> q$, $\mathbf{M}$ approximately can be expressed as

$$\mathbf{M} = \min(N-m, m, l, N-l)\mathbf{E}. \qquad (33)$$

where $\mathbf{E}$ is a square matrix and all the elements of $\mathbf{E}$ are equal to 1, thus

$$\begin{aligned} E(\varepsilon^2) &\approx \frac{144}{\mathrm{A}^2N^2(N^2-1)^2}\sum_{m=\frac{N+1}{2}}^{N-1}\sum_{l=\frac{N+1}{2}}^{N-1} \qquad (34) \\ &\quad \min(N-m, N-l)\mathbf{1}^T\mathbf{b}\mathbf{b}^T\mathbf{1} \\ &= \frac{6\sigma^2|(1 + b_1e^{j\omega_0} + b_2e^{j2\omega_0} .... + b_qe^{jq\omega_0})|^2}{\mathrm{A}^2N(N^2-1)} \\ &= \frac{6\psi(\omega_0)}{\mathrm{SNR}N^2(N^2-1)} \end{aligned}$$

This is the asymptotic CRB of frequency estimation in the presence of colored noise given in [14], where $\psi(\omega)$ is the power spectral density of colored noise as given in (21). (The reason of being interested in asymptotic CRB lies in the simplicity of such a formulae compared with the complexity and lack of insights corresponding to the finite sample exact CRB. Another reason is its good accuracy for large data samples as shown in [9, 14].) Thus we have shown that the proposed estimator asymptotically achieves CRB in MA noise. In Fig.2, we show the estimation of normalized frequency at $f = 0.124$, corrupted by colored noise. The noise is generated by passing AWGN through a MA filter with system transfer function $B(z) = 1 - 1.8z^{-1} + 1.6z^{-2} - 2.4z^{-3}$. It can be seen that the variance of the estimator agrees with the asymptotic CRB. The exact CRB is also shown in crossed line for comparison. It is also noted that the MSE of this frequency estimator in MA colored noise is insensitive to the order of the MA process if $q << N/2$, as only lags ($\frac{N+1}{2}$, $\frac{N+3}{2}$...$N-1$) are used in the estimation.

4. CONCLUSION

We propose a novel frequency estimator based on the multiple lags of autocorrelation in conjunction with DFT based phase unwrapping. It is shown that this estimator is efficient even for short length of data samples while maintaining a low SNR threshold. The performance of the estimator with the optimal window is derived and the relation with MLE is explored. It is shown that this estimator is statistically similar to MLE but avoids the exhaustive fine grid search of the MLE. Furthermore, this estimator is used in frequency estimation in MA colored noise. It is proven that the variance of the estimator achieves the asymptotic CRB. The performance of the estimator is verified by simulation.

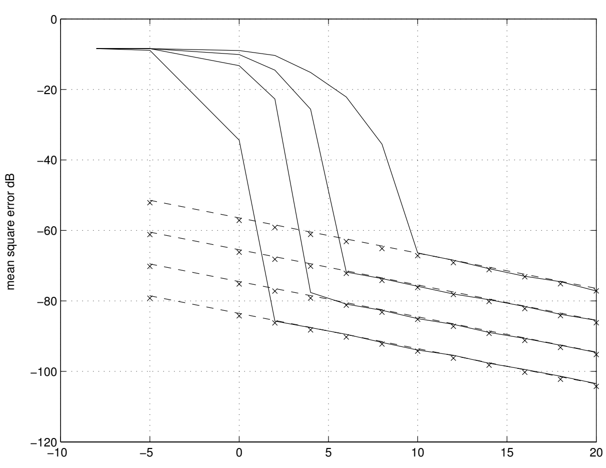

Fig. 2. The MSE of the proposed estimator in MA colored noise compared with the asymptotic CRB and exact CRB, $f = 0.124$, $N = 64, 128, 256, 512$

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