PARAUNITARY FILTER BANK DESIGN VIA A POLYNOMIAL SINGULAR-VALUE DECOMPOSITION

Soydan Redif and Tom Cooper *

Advanced Signal & Information Processing Group, QinetiQ Ltd., Malvern Technology Centre, St Andrews Rd, Malvern, Worcestershire WR14 3PS. Fax: 01684 896502, Phone: 01684 896648, Email: s.redif@signal.qinetiq.com

ABSTRACT

We propose a modification to a polynomial SVD technique, known as SBR2, which enables it to be applied effectively to the task of optimal FIR paraunitary filter bank design for use in subband coding. We compare our technique, namely the SBR2 coder, to a state-of-the-art FIR compaction filter design method. Here, we show that, in the particular case of a two-channel filter bank, high coding gains are obtainable with our technique for a small number of algorithm iterations.

1. INTRODUCTION

The design of filter banks (Fig. 1) for subband coding of signals has been extensively studied [1]-[5]. For the case where the order of the filters is unconstrained, it is known that a *principal component filter bank* (PCFB) exists and is an optimal orthonormal (paraunitary) filter bank for this problem [2], [3]. This is also true when the filter orders are constrained to be not greater than the number of subband channels. In this case, the Karhunen-Loeve transform (KLT) or the singular-value decomposition (SVD) provide the optimal solution. It has been shown that, in general, the PCFB does not exist for the intermediate case where order-constrained filters (or finite impulse response (FIR) filters) are used [4]. This is except for the special case of the two-channel filter bank.

A number of authors have proposed methods for the design of suboptimal (near-optimal) constrained-order orthonormal filter banks [5], [6]. A technique that achieves this efficiently is the window method, proposed in [5]. This technique can be used to design a compaction filter for a signal with a given power spectral density (psd). A two-channel FIR paraunitary filter bank is easily constructed using this method; a strategy for multichannel orthonormal filter bank design has not been published. In [6], Moulin *et al* present an alternative approach which uses linear programming to design paraunitary filter banks. The algorithm is computationally costly and complex for large filter orders. In contrast, the window method is less complex and faster even for large filter orders.

Other authors have presented paraunitary filter bank design methods in the context of signal subspace analysis of broadband signals [7], [8]. The approach by Regalia and Loubaton in [7] exploits the fixed degree parameterisation proposed by Vaidyanathan [1]. They re-formulate the problem using a state space model and propose an iterative solution, which avoids the problems of local minima associated with gradient descent techniques. An extension of the SVD to broadband signals is proposed in [8]. The technique essentially performs a SVD (by way of the QR-algorithm) of a set of *windowed* data at every frequency. This algorithm is capable of producing accurate subspace decomposition.

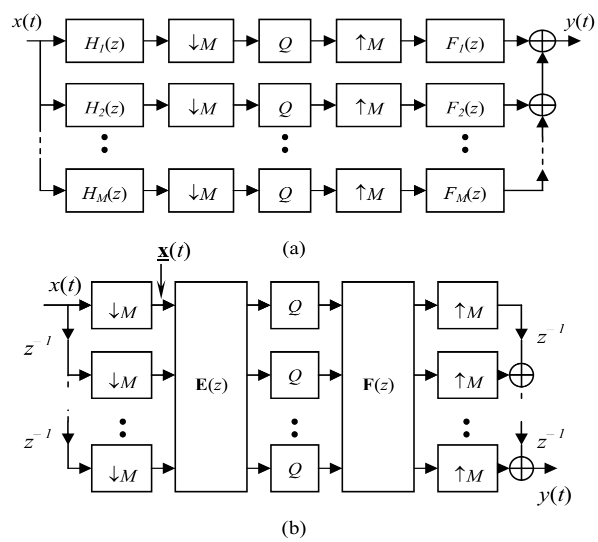


Figure 1 (a) M–channel uniform maximally decimated filter bank and (b) its polyphase representation.

Another method that can be viewed as a generalisation of the SVD to broadband signals (or polynomial matrices) is proposed in [9]. This algorithm is called the second order sequential best rotation (SBR2) algorithm. The SBR2 algorithm has been successfully used in applications where the SVD has traditionally been employed; including subspace decomposition and multichannel data compression. In this paper, we present an adaptation of this technique for the purpose of designing an orthonormal filter bank for subband coding.

2. FILTER BANK OPTIMALITY

An M–channel subband coder is shown in Fig. 1(a) and its polyphase form is shown in Fig. 1(b). This is a maximally

* This work has been sponsored by the United Kingdom Ministry of Defence Corporate Research Programme.

decimated uniform filter bank; our discussions are limited to this type of subband coder. We further assume that the filter bank is orthonormal, i.e., $\mathbf{E}(e^{j\omega})$ is unitary for all ω. In other words, the matrix $\mathbf{E}(z)$ is paraunitary ($\mathbf{E}(z)\widetilde{\mathbf{E}}(z)=\mathbf{I}$), where $\mathbf{E}(z)$ is a matrix of polynomials (or polynomial matrix) [1]. If $\mathbf{F}(z)$ is chosen such that $\mathbf{E}(z)\mathbf{F}(z)=cz^{-\tau}\mathbf{I}$, for some constant c and integer n, then the subband coder is a perfect reconstruction filter bank. That is, with no subband-processing, $y(t)=x(t-\tau)$ for all t and τ.

A PCFB offers an optimal solution to two subband coding problems. Firstly, it is an optimal orthonormal subband coder in the sense of maximising the *coding gain* [2]

$$G=\frac{(1/M)\sum_{\ell=1}^{M}\sigma_{v_\ell}^2}{\left(\prod_{\ell=1}^{M}\sigma_{v_\ell}^2\right)^{1/M}}, \qquad (1)$$

where, $\sigma_{v_\ell}^2$ is the variance of $v_\ell(t)$: the output of $H_\ell(z)$. Secondly, it minimises the reconstruction error for a proper subset of the set of subband channels. Vaidyanathan has shown that the outputs of a PCFB simultaneously satisfy:

Strong decorrelation. The subband signals, $v_\ell(t)$, are decorrelated at all relative time lags, i.e.,

$$E[\sum_t v_\ell(t)v_m(t+\tau)]=0, \qquad (2)$$

for $\ell \neq m$ and all τ.

Spectral majorisation. Let the psd of $v_\ell(t)$ be denoted as $S_{\ell\ell}(e^{j\omega})$. The set $\{S_{\ell\ell}(e^{j\omega})\}$ has the spectral majorisation property if, for all ω,

$$S_{11}(e^{j\omega})\geq S_{22}(e^{j\omega})\geq\ldots\geq S_{MM}(e^{j\omega}), \qquad (3)$$

where the subbands are numbered such that $\sigma_{v_\ell}^2 \geq \sigma_{v_{\ell+1}}^2$. however, in general Eq. (1) does not represent the objective.

3. POLYNOMIAL MATRIX SVD

Correlation between signals is a type of redundancy which can be exploited to achieve compression. If the signals are correlated at a single relative time-lag then the KLT (or SVD) can be used to (instantaneously) decorrelate the signals. The decorrelation process converts the form of the redundancy from correlation between the signals to disparity between the signal powers. At this stage, it is possible to achieve compression by discarding low power channels. Viewed differently, the algorithm estimates a noise subspace, which may be discarded. (Alternatively, the low power channels may be encoded with a lower number of bits than the dominant channels.)

In the case of the KLT/SVD, the matrix $\mathbf{E}(z)$ in Fig. 1(b) is a unitary matrix, which is applied to the vector $\underline{\mathbf{x}}(t)$ (or polynomial vector $\underline{X}(z)$, which contains 'blocked' samples of the input signal $x(t)$). The orthogonality condition implies that the transformation is energy preserving, so the algorithm can be used to identify signal-plus-noise and noise subspaces.

However, if $\underline{\mathbf{x}}(t)$ contains signals that are correlated at more than one time-lag, instantaneous decorrelation by a unitary matrix is no longer sufficient for maximisation of Eq. (1); strong decorrelation (or decorrelation at all relative time lags) is necessary. For accurate estimation of the broadband signal subspace, strong decorrelation must be imposed. To achieve this, the transformation applied is required to be in the form of a matrix of polynomials, or equivalently a bank of FIR filters. Such a filter bank can be found by the SBR2 algorithm.

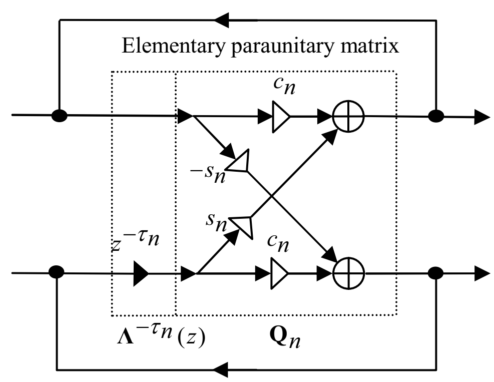
The SBR2 algorithm [9] uses a simple scheme for generating polynomial (FIR) paraunitary matrices for the strong decorrelation of multiple channels. A paraunitary matrix represents an all-pass filter bank and, accordingly, it preserves the total signal energy at every frequency [1]. The structure of the filter bank produced by the technique is an immediate generalisation of the paraunitary matrix decomposition found by Vaidyanathan in [1]. For the 2 x 2 case, the paraunitary matrix may be expressed as,

$$\mathbf{H}(z)=\mathbf{Q}_L\mathbf{\Lambda}^{\tau_L}(z)\ldots\mathbf{Q}_0\mathbf{\Lambda}^{\tau_0}(z). \qquad (4)$$

Here, the polynomial matrix $\mathbf{\Lambda}^{\tau_n}(z)$ has the form $\mathbf{\Lambda}^{\tau_n}(z)=\begin{bmatrix}1 & 0\\ 0 & z^{\tau_n}\end{bmatrix}$, where the integer parameters τ_n can be negative or positive. The matrix $\mathbf{Q}_n$ is a 2×2 unitary matrix and $\mathbf{Q}_n\mathbf{\Lambda}^{\tau_n}(z)$ is an elementary paraunitary matrix/block. The SBR2 is an iterative algorithm. At each step of the algorithm, the signals produced by the previous step are used to find an optimal time-delay and a rotation matrix, which are then applied to those signals. This is repeated a predefined number times. A graphical representation of this decomposition is provided in Fig. 2. The polynomial matrix generated by Eq. (4) is paraunitary since each stage is paraunitary. Note that the degree is not specified beforehand.

The parameter values for each block are chosen with a greedy optimisation scheme; a generalisation of the classical Jacobi algorithm is employed. SBR2 has recently been developed into a multichannel algorithm that performs, to a good approximation, strong decorrelation and spectral majorisation.

The algorithm can be classed as a blind technique since its formulation is not based on knowledge of the true statistics of the input signal. Therefore, the performance of the filter bank it applies depends on the accuracy of its estimate of the true covariance matrix for input signals.



where $c_n=\cos\theta_n, s_n=\sin\theta_n, \theta_n\in\mathbf{R}$

Figure 2 Flow diagram of SBR2

The SBR2 algorithm can be used directly to construct an $M \times M$ paraunitary polynomial matrix $\mathbf{E}(z)$ for the decimated subband signals $\underline{X}(z)$ in Fig. 1. The output subband signals from $\mathbf{E}(z)$ may be expressed as $\underline{V}(z) = \mathbf{E}(z)\underline{X}(z)$. However, if the input signal $x(t)$ is stationary, this scheme can be improved upon. In this case, the statistics of the demultiplexed input signal have some structure. This knowledge is implicitly exploited by conventional filter bank design algorithms, such as the window method.

4. SBR2 CODER

As discussed above, applying the matrix $\mathbf{E}(z)$ constructed by SBR2 directly to $\underline{X}(z)$ does not use all the information available. The true covariance matrix, $\mathbf{A}(z)$, of the outputs, $\underline{X}(z)$, of the demultiplexer in Fig. 1(b) has a pseudocirculant structure (defined below). The coefficients of the polynomial entries of $\mathbf{A}(z)$ are correlations between pairs of signals at different time-lags. In this paper, we say that $\mathbf{A}(z)$ is a *polynomial covariance matrix*. A polynomial covariance matrix is usually associated with broadband or convolutively mixed signals. Since the SBR2 algorithm can operate on arbitrary input signals, the pseudocirculant structure is ignored. For the effective application of SBR2 to coding, the algorithm needs to be modified.

4.1. Pseudocirculant matrices

An M x M polynomial matrix $\mathbf{A}(z)$ with entries $A_{\ell,m}(z)$ is said to be pseudocirculant if there exist polynomials $\Phi_0(z), \Phi_1(z), \ldots, \Phi_{M-1}(z)$ such that

$$A_{\ell,m}(z) = \begin{cases} \Phi_{m-\ell}(z), & 1 \le \ell \le m \le M \\ z^{-1}\Phi_{m-\ell+N}(z), & 1 \le m < \ell \le M. \end{cases} \tag{5}$$

In words, $\mathbf{A}(z)$ is a circulant matrix except that the entries below the main diagonal are multiplied by z^{-1}.

4.2. Wide-sense stationarity

A stochastic process is said to be wide-sense stationary (WSS) if and only if: (1) $E[x(t)] = E[x(t + \tau)]$, for all integers t and τ, and (2)

$$E[x(t)x\,(t+\tau)] = a(\tau), \quad \text{for all } \tau, \tag{6}$$

where $a(\tau)$ is the autocovariance function of $x(t)$. Note that we assume $E[x(t)] = 0$ for all t.

4.3. The structure of the covariance matrices

We show that the covariance matrix of a demultiplexed WSS signal has a pseudocirculant structure. The blocked samples from the demultiplexer in Fig. 1 are $\underline{\mathbf{x}}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, where $x_k(t) = x(Mt + k - 1)$ ($1 \le k \le M$). A typical term from the true covariance matrix of $\underline{x}(t)$, $\mathbf{A}(z)$, is (ignoring normalisation)

$$\begin{aligned} A_{l,m}(z) &\propto E\left[\left(\sum_t x_l(t)z^{-t}\right)\left(\sum_t x_m(t)z^{t}\right)\right] \\ &= E\left[\left(\sum_t x(Mt+\ell-1)z^{-t}\right)\left(\sum_t x(Mt+m-1)z^{t}\right)\right] \\ &= \sum_\tau E\left[\sum_t x(M(t+\tau)+\ell-1)x(Mt+m-1)\right]z^{-\tau} \\ &\propto \sum_\tau a([M(t+\tau)+\ell-1]-[Mt+m-1])z^{-\tau} \\ &= \sum_\tau a(M\tau+m-\ell)z^{-\tau}. \end{aligned}$$

So, setting $\Phi_k(z) = \sum_\tau a(M\tau + k - 1)z^{-\tau}$, we see that $\mathbf{A}(z)$ is pseudocirculant.

4.4. Exploiting signal statistics

The SBR2 algorithm can be modified to exploit the pseudocirculant structure of $\mathbf{A}(z)$. The set of diagonally related elements of $\mathbf{R}(z)$ are different estimates of the same true cross-covariance. Therefore, to improve the estimate of $\mathbf{A}(z)$, averaging may be performed across the associated coefficients in $\mathbf{R}(z)$ – taking account of the delay between terms above the diagonal and those below.

We define

$$\varphi_k(\tau) = \frac{1}{M}\left(\sum_{\ell=1}^{M-k} r_{\ell,\ell+k}(\tau) + \sum_{\ell=M-k+1}^{M} r_{\ell,\ell+k-M}(\tau+1)\right), \tag{7}$$

$0 \le k \le M-1$, and a typical entry of the new (averaged) sample covariance matrix $\mathbf{R}'(z)$ as

$$R'_{\ell,m}(z) = \begin{cases} \sum_\tau \varphi_{m-\ell}(\tau)z^{-\tau}, & 1 \le \ell \le m \le M \\ \sum_\tau \varphi_{m-\ell+M}(\tau)z^{-\tau-1}, & 1 \le m < \ell \le M. \end{cases} \tag{8}$$

The operations that the SBR2 algorithm performs are determined by $\mathbf{R}(z)$. It is straightforward to modify the algorithm to 'work in the covariance domain' and operate on the covariance matrix directly. The algorithm can then be applied to our improved estimate of the covariance matrix. This modification yields the *SBR2 coder*.

5. SIMULATIONS

We present simulation results relating to an investigation into the coding gain performance of the SBR2 coder. The results are compared to the window method, which was chosen because of its simplicity and efficiency. We modelled the two-channel filter bank (Fig. 1 for M = 2) and used the SBR2 coder and the window method to design appropriate analysis banks for an input signal $x(t)$. This signal was obtained from the output of an order 5 autoregressive (AR(5)) filter with the transfer function $G(z) = U(z)/W(z)$, where $U(z) = 0.6903 - 0.0160z^{-1} - 0.1453z^{-2} + 0.3302z^{-3} - 0.5426z^{-4} - 0.3141z^{-5}$ and $W(z) = 0.6867 - 0.4363z^{-1} + 0.1255z^{-2} - 0.3162z^{-3} + 0.4688z^{-4} - 0.0516z^{-5}$. The AR(5) filter was driven by samples drawn from a white noise process of zero mean and unit variance. An AR model is regarded as a good model for many practical signals such as image and speech signals. The true psd of $x(t)$ is shown in Fig. 3.

The magnitude-squared frequency response of the filters $H_1(z)$ and $H_2(z)$ constructed by the SBR2 coder after 100 iterations are shown as the solid and dotted curves in Fig. 3, respectively. It is clear from Fig. 3 that the SBR2 coder has designed a multiband compaction filter with passbands that coincide with dominant signal frequency components. This is indicative of a high coding gain.

The coding gain performances of the SBR2 coder and the window method are shown in Fig. 4. In the case of SBR2, the Figure shows coding gain versus the number of algorithm iterations/steps or delay-rotation stages. In case the of the window method, the coding gain was calculated for a number of *fixed* filter orders. The straight dotted line represents the maximum attainable (ideal) coding gain $G' = 1.94$. The performance of both filter banks approaches the ideal performance as the number of iterations and the filter order increases. The window method produces marginally better coding gain performance for high filter orders than the SBR2 coder does for a high number of steps. However, the SBR2 coder achieves high coding gain for a small number of steps – in the range (1,~50). This is because SBR2 has the freedom to choose the most important filter coefficients first, whereas, the window method has a fixed order filter, which it must parameterise. The simplicity of the window method makes it computationally more efficient than the SBR2 coder. In order to represent the filter designed by SBR2 one real number and one (typically small) integer are required for each step. By contrast, the filter bank designed using the window method requires one real number for each filter tap.

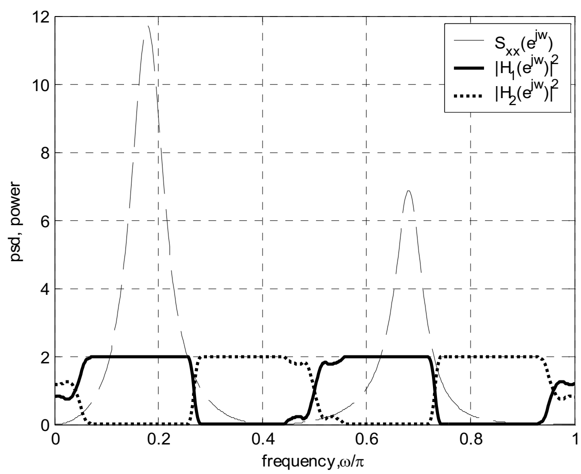


Figure 3 Power spectral density of an AR(5) process and the frequency response of the filters designed by the SBR2 coder.

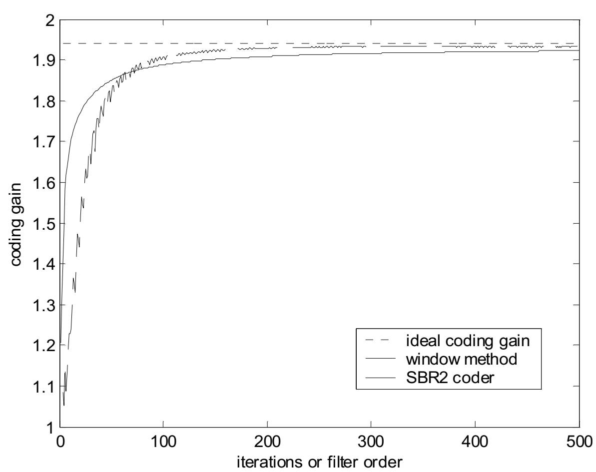


Figure 4 Coding gain versus the number of iterations (or filter order) for the SBR2 coder and the window method.

6. CONCLUSIONS

In this paper, we have presented a new algorithm, namely the SBR2 coder, for the design of paraunitary filter banks. Its performance has been characterised and compared to that of the window method. In the particular case of a two-channel filter bank, the SBR2 coder attains high coding gains for a small number of iterations (short filters). For large filter orders, the window method performs slightly better than the SBR2 coder.

The SBR2 algorithm, unlike the window method, does not assume that signals have stationary statistics; therefore, it could be applied much more readily to non-stationary signals. The non-stationarity condition is found in a number of applications, such as, practical communication systems. Another potential benefit of using the SBR2 coder arises in situations where perfect reconstruction is required for the multichannel case. The SBR2 coder produces strictly paraunitary polyphase matrices. In applications where lossless compression is required the SBR2 coder may be a more suitable algorithm than a multichannel window method; where exact paraunitariness will not hold.

It is envisaged that the SBR2 coder can also be extended naturally to the case of multiple input signals, i.e., it can be used in a multi-input, multi-output subband coding scheme. This is left for future exploration.†

7. REFERENCES

[1] Vaidyanathan P.P., *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ: Prentice-Hall 1993.

[2] Vaidyanathan P.P., "Theory of optimal orthonormal filter banks", *IEEE Proc. ICASSP-96*, vol. 3, pp. 1487-1490, Atlanta, May 1996.

[3] Tsatsanis M.K. and Giannakis G.B., "Principal component filter banks for optimal multiresolution analysis", *IEEE Trans. on Signal Processing*, Vol. 43, No. 8, August 1995.

[4] Kirac A. and Vaidyanathan P.P., "On existence of FIR principal component filter banks", *IEEE Int. Conf. Asp*, Seattle, 1998.

[5] Kirac A. and Vaidyanathan P.P., "Theory and design of optimum FIR compaction filters", *IEEE Trans. on Signal Process.*, Vol. 46, No. 4, April 1998.

[6] Moulin P. and Mihcak M.K., "Theory and design of signal-adapted FIR paraunitary filter banks", *IEEE Trans. on Signal Process.*, Vol. 46, No. 4, April 1998.

[7] Regalia P.A. and Loubaton P., "Rational subspace estimation using adaptive lossless filters", *IEEE Trans. Signal Process.*, Vol. 40, No. 10, pp. 2393-2405, Oct 1992.

[8] Lambert R.H., Joho M., and Mathis H., "Polynomial singular values for number of wideband source estimation and principal component analysis", http://ica2001.ucsd.edu/index_files/pdfs, 2002.

[9] McWhirter J.G. and Baxter P.D., "A novel technique for broadband SVD", The twelfth adaptive sensor array processing workshop, MIT Lincoln Laboratory, March 2004.

† The authors are grateful to John McWhirter of QinetiQ Ltd and Stephan Weiss of the University of Southampton for their guidance and support.