STRUCTURALLY LINEAR PHASE FACTORIZATION OF 2-CHANNEL FILTER BANKS BASED ON LIFTING

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ABSTRACT

Lifting is advantageous for its structurally perfect reconstruction property of a filter bank. Based on such lifting structure, this paper is focused on factorization of 2-channel linear phase perfect reconstruction filter bank. We propose factorizations that are also structurally linear phase for this kind of filter bank. For different types of filter banks, we use different approaches to achieve it. We apply the factorization to existing filter banks, and illustrate its advantage towards efficient implementation and reduced precision implementation.

1. INTRODUCTION

Perfect-reconstruction (PR) filter banks (FB) with linear phase (LP) have been extensively used in fields of image and signal processing. In [1] Nguyen et al offered many important guidelines in 2-channel LPPRFB and developed the lattice structures. However their solution for type-B is redundant. Since then many other approaches have been proposed, such as [2]. The structure proposed in [2] is novel in the sense that it employed the Chebyschev representation of the polynomials to factorize a FB. But for a given analysis bank constructed by their method, its inverse-which though may have a similar form-has to be recalculated. Besides, the scaling in the inverse structure may be totally different from its counterpart and expensive to implement.

Lifting structure in Figure 1, first known as ladder structure, was adapted for wavelet factorization by Sweldens et al in [3]. Since the FB continues to retain PR even when the lifting coefficients are quantized, we may achieve the structurally PR property for both design and implementation. Due to such advantage, the lifting structure has been explored extensively for FB's. The problem with the lifting factorization of [3] is its high non-uniqueness leading to many inefficient solutions. Take the factorization of Bior2.2 (a biorthogonal wavelet in MATLAB) for an example. The following factorizations are all legal results from the Euclidean Algorithm in [3].

$$\mathbf{E}(z)=\begin{bmatrix}\frac{7\sqrt{2}}{8} & 0\\ 0 & -\frac{8}{7\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & \frac{16}{49}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{7}{16}z^{-1} & 1\end{bmatrix}\begin{bmatrix}1 & \frac{2}{7}z\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -0.5 & 1\end{bmatrix}$$

$$\mathbf{E}(z)=\begin{bmatrix}1 & 0\\ \frac{2}{7} & 1\end{bmatrix}\begin{bmatrix}\frac{7\sqrt{2}}{8} & 0\\ 0 & -\frac{8}{7\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & \frac{16}{7}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{z}{16} & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{-z^{-1}}{2} & 1\end{bmatrix}$$

Above structures are not satisfactory in more than one way. First, when implementing the structure with finite precision, the LP property is no longer guaranteed. Therefore these structure are not structurally LP. Further, these structures are not efficient since the number of coefficients are not less than that of the direct form. In this work we aim to find a lifting factorization that is both structurally PR and structurally LP, unique and efficient.

	Type-A	Type-B
length	$(2N_0, 2N_1)$	$(2N_0+1, 2N_1+1)$
symmetry	sym. & antisym.	both are sym.
length combination	both N_0 and N_1 are even or odd	N_0 is even and N_1 is odd or vice versa
Degree of freedom	$\frac{N_0+N_1}{2}$	$\frac{N_0+N_1+3}{2}$
Distance of sym. centers	even	odd

Table 1. The properties for 2-channel LPPRFB

A complete approach is presented in this paper for the factorization of 2-Channel LPPRFB. To achieve structurally LP, each lifting block is made LP (symmetric). For 2-channel LPPRFB, the low pass and high pass filters are symmetric and related to each other due to the PR property. We exploit the symmetry in the form of PR condition and prove that we can enforce such symmetry to each lifting block. Further, explicit expression for the lifting blocks are provided. The proposed structure has many desirable properties inheriting from lifting.

1.1. Preliminary

Given a 2-channel FIR FB with analysis polyphase matrix $\mathbf{E}(z)$, we have PR property, if and only if [4]

$$det(\mathbf{E}(z)) = \alpha \cdot z^{-n_0} \tag{1}$$

where n_0 is an integer (which can be non-negative for causal filters). There are only two types of 2-channel LPPRFB, type-A and type-B, as defined in [1]. We summarize some properties of these two types from [1] in Table 1, where N_0 and N_1 are the numbers of (anti)symmetric coefficients for the analysis filters $H_0(z)$ and $H_1(z)$ respectively.

Note that we have two kinds of operation that can move the symmetry centers without destroying PR. They are (a) inserting the same amount of delays to both channels, and (b) inserting even number of delays to either channel. We will use them to simplify our description and derivation.

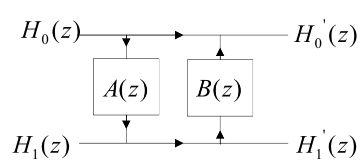

Fig. 1. Lifting structure

2. LIFTING FACTORIZATION OF TYPE-B FB

Express an arbitrary type-B FB as

$$\begin{aligned} H_0(z) &= z^{c_0}\left(\sum_{n=1}^{N_0} h_0(n)(z^{-n}+z^{n}) + h_0(0)\right) \\ H_1(z) &= z^{c_1}\left(\sum_{n=1}^{N_1} h_1(n)(z^{-n}+z^{n}) + h_1(0)\right) \end{aligned} \quad (2)$$

With two operations defined above their symmetry centers can be moved to $c_0 = 0$ and $c_1 = -1$ respectively. For this given FB its PR condition (1) is actually comprised of a set of equations and can be expressed in the following compact form.

$$\begin{aligned} &(-1)^{N_0}\sum_{k=0}^{2i-1}(-1)^k h_0(|N_0-k|)h_1(|N_1-2i+1+k|) \\ &= \alpha\delta(i-\tfrac{N_0+N_1+1}{2})\ ,\ i=1,2,\cdots\tfrac{N_0+N_1+1}{2} \end{aligned} \quad (3)$$

Now we explain the factorization steps as follows.

2.1. The procedure of factorization

In a cascading structure, it is desirable (e.g. for the sake of adaptation) that each block retains some basic properties of the FB such as symmetry. In fact PR conditions in (3) guarantee the existence of such cascading structure and determine the corresponding lifting structure uniquely. We also employ the symmetry nature of the FB to the most extent and find an LP structure for every cascading block. This achieves the structurally LP property. Now we show how to explicitly find the lifting structure.

1) Without loss of generality, let $N_0 > N_1$ and N_0 be an even number. We use the long division of polynomial as follows.

$$\begin{bmatrix} H_0'(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & B(z) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}$$

Taking into account the symmetry nature of filters the quotient polynomial can be chosen as $B(z) = bz(z^p + z^{-p})$ and $b = -\frac{h_0(N_0)}{h_1(N_1)}$, where $p = N_0 - N_1$ is the half-length difference of two filters. In this way we can reduce the coefficients of $z^{\pm N_0}$ in $H_0'(z)$ to zero. It's more interesting that simultaneously this choice can also bring us the elimination of the items with $z^{\pm(N_0-1)}$ in $H_0'(z)$. In fact the bonus relies on the first equation of (3)

$$h_0(N_0)h_1(N_1-1) - h_0(N_0-1)h_1(N_1) = 0 \quad (4)$$

Therefore the length of $H_0'(z)$ is $2N_0 - 3$.

2) Repeat step 1) until

$$N_0' = N_1 + 1 \quad (5)$$

For every iteration we can always reduce the longer filter by 4 taps. This is guaranteed by the relating constraint in (3).

3) Now dual lifting and lifting matrices can be used as a pair. We multiply them to $[\hat{H}_0(z), \hat{H}_1(z)]$:

$$\begin{bmatrix} \hat{H}_0'(z) \\ \hat{H}_1'(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ A(z) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & B(z) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{H}_0(z) \\ \hat{H}_1(z) \end{bmatrix}$$

To retain the symmetry as well as the location of the symmetry centers we choose $A(z) = a(1+z^{-2})$ and $B(z) = b(1+z^2)$. Here $b = -\frac{\hat{h}_0(N_1+1)}{\hat{h}_1(N_1)}$ and

$$a = \frac{-\hat{h}_1(N_1)\hat{h}_1(N_1)}{\hat{h}_0(N_1-1)\hat{h}_1(N_1) - \hat{h}_0(N_1+1)\left(\hat{h}_1(N_1-2)+\hat{h}_1(N_1)\right)}$$

With such choice we can reduce coefficients of $z^{\pm(N_1+1)}$ and $z^{\pm N_1}$ in $\hat{H}_0'(z)$ to zero. For $\hat{H}_1'(z)$ it's $z^{\pm(N_1)}$ and $z^{\pm(N_1-1)}$ that we cancel together.

4) After step 3) the length of $\hat{H}_0'(z)$ and $\hat{H}_1'(z)$ is $2N_1 - 1$ and $2N_1 - 3$ respectively. This is similar to the relation in (5). Hence we can perform step 3) again. We recur this step downward until the length of $\hat{H}_0'(z)$ and $\hat{H}_1'(z)$ are 5 ($\Leftrightarrow N_0 = 2$) and 3 ($\Leftrightarrow N_1 = 1$) respectively. We denote them as follows.

$$\begin{aligned} H_0^{\triangle}(z) &= h_0^{\triangle}(2)(z^2+z^{-2}) + h_0^{\triangle}(1)(z^1+z^{-1}) + h_0^{\triangle}(0) \\ H_1^{\triangle}(z) &= h_1^{\triangle}(1)(z^{-2}+1) + h_1^{\triangle}(0)z^{-1} \end{aligned}$$

Considering the last two PR equations in (3), we can verify the following equation,

$$\begin{bmatrix} 1 & 0 \\ \frac{-h_1^{\triangle}(1)\cdot(1+z^{-2})}{h_0^{\triangle}(0)-2h_0^{\triangle}(2)} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{h_0^{\triangle}(2)\cdot(1+z^2)}{h_1^{\triangle}(1)} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H_0^{\triangle}(z) \\ H_1^{\triangle}(z) \end{bmatrix} = \begin{bmatrix} \alpha(h_1^{\triangle}(0))^{-1} \\ h_1^{\triangle}(0)\cdot z^{-1} \end{bmatrix}$$

where α is the same one as in (1). This relation gives us a perfect end to the factorization.

Now we present the complete form of the above factorization for a given type-B FB. Numbering the $A(z)$'s and $B(z)$'s inversely (i.e. $A_1(z)$ is for the last step, etc.), and denoting $h_1^{\triangle}(0)$ as K the following can be obtained.

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \left(\prod_{n=1}^{\frac{N_1+1}{2}} \begin{bmatrix} 1 & -B_n(z) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -A_n(z) & 1 \end{bmatrix}\right)\begin{bmatrix} \alpha K^{-1} \\ Kz^{-1} \end{bmatrix}$$

Here $A_n(z) = a_n(1+z^{-2})$ when $n \le \frac{N_1+1}{2}$ and $B_n(z) = b_n(1+z^2)$ when $n < \frac{N_1+1}{2}$. Note that

$$\begin{pmatrix} 1 & B_1(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & B_2(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & B_1(z)+B_2(z) \\ 0 & 1 \end{pmatrix}.$$

and by combining all $B(z)$ in 2) and the first $B(z)$ in 3) we obtain

$$B_{\frac{N_1+1}{2}}(z) = b^{(1)}_{\frac{N_1+1}{2}}(1+z^2) + \cdots + b^{(\frac{p+1}{2})}_{\frac{N_1+1}{2}}(z^{-p+1}+z^{p+1})$$

Figure 2 shows the corresponding polyphase form of analysis FB (we simplify the figure by assuming $N_0 = N_1 + 1$ and $\alpha = 1$). Its synthesis counterpart can be achieved simply by changing the sign of every lifting block and mirror flipping them. It's easy to extend such factorizations to cases with either $N_0 < N_1$ or N_0 to be odd number.

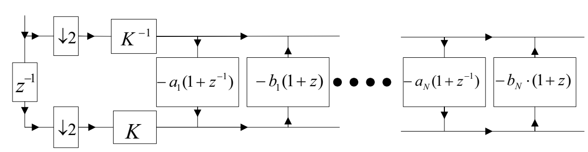

Fig. 2. The type-B analysis filter bank with lifting

Note that the factorization of [3] may also lead to a structurally LP form, such as some examples given in [3]. However, there is no systematic approach to reach such factorization in [3], while the proposed factorization is general and explicit. We can declare that given an arbitrary type-B (not only some special wavelets) FB we can always factorize it into a succinct manner. With these factorizations we impose the LP property to the structure just as what we have done to PR property using lifting. We also obtain some other benefits. This will be illustrated in section 4.

3. LIFTING FACTORIZATION OF TYPE-A FB

Similarly, for an arbitrary type-A FB with $H_0(z)$ and $H_1(z)$, their symmetry centers are first aligned to the same point $z^{-\frac{1}{2}}$. This is always feasible according to Table 1. They are expressed as

$$\begin{aligned} H_0(z) &= \left(\sum_{n=0}^{N_0-1} h_0(n)(z^{-n-1} + z^n)\right) \\ H_1(z) &= \left(\sum_{n=0}^{N_1-1} h_1(n)(z^{-n-1} - z^n)\right) \end{aligned} \tag{6}$$

Still according to the Table 1, the length difference of two filters is a multiple of 4. Using similar technique we can reduce the length of the longer filter by 4 taps in every iteration of lifting factorization. This is guaranteed by the PR condition. So we can always reduce the longer channel up to the length of the shorter channel. The procedure to do this is quite similar to that of type-B. For the sake of brevity, we don't repeat it. When we reduce them to the same length ($N_0 = N_1$ in (6)), such factorization can no longer be extended. It's because the lifting factorization for type-B will reduce the filter length alternatively, and the length difference of two channels enables this reduction to continue. For type-A the difference may vanish (impossible for type-B). In this case the previous factorization becomes ineffective.

At this stage, we have two choices. One is to use the Euclidean Algorithm as in [3]. As we mentioned, Euclidean Algorithm doesn't provide any explicit choice of quotient polynomials. As depicted in Fig. 3, we propose a lifting factorization that employs the symmetry and has a cascading manner. In order to enforce the LP property to the structure, we choose $A_i = \frac{h_0(N-1)}{h_0(N-2)}$, $B_i = \frac{h_0(N-1)h_0(N-2)}{h_0^2(N-2)-h_0^2(N-1)}$ and $k_i = \frac{h_0^2(N-2)}{h_0^2(N-2)-h_0^2(N-1)}$ for the i-th block, where $h_0(N-1)$ and $h_0(N-2)$ are the first two impulse response coefficients (in the notation of (6)) of the low pass filter $H_0(z)$ of length $2N$ up to i-th block. Dual lifting block $F(z)$ takes care of the unequal part of the FB and can also be uniquely determined by the given FB.

The other choice we have is we can directly use the lattice structure for type-A proposed in [1]. For the FB with filters of equal-length, the lattice structure is quite effective in the sense that it can structurally impose the LP as well as PR (up to a scaling) property and number of coefficients are canonical.

It may be shown that the proposed scheme in Fig. 3 is equivalent to the lattice scheme. However, the former scheme retains the structurally LP in the field of real and rational numbers while the lattice retains LP even for integers. Consequently, the efficient representation of a type-A FB may be a hybrid structure. For unequal-length part we use lifting structure, and for the remaining part we use lattice structure. We illustrate this in section 4.

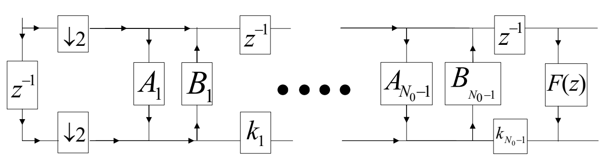

Fig. 3. The type-A analysis filter bank with lifting

4. EXAMPLES AND APPLICATIONS

4.1. Factorization examples

We apply the proposed factorization to the following FBs: The FB example used in section 1 (FB1); Popular CDF 9/7 wavelet (FB2); 2 FBs from [5], where they are regarded as wavelets for image compression (FB3 & FB4). The coefficients for first two FBs are easily available, hence we only list the effective coefficients for last two FBs in Table 2. In the following $\gamma(z) = 1 + z$ and Table 3 gives the lifting coefficients.

FB1: $\mathbf{E}(z) = \begin{bmatrix} 1 & -\frac{1}{2}\gamma(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{4}\gamma(z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

FB2: $\mathbf{E}(z) = \left(\prod_{i=1}^{2} \begin{bmatrix} 1 & a_{2i-1}\gamma(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_{2i}\gamma(z^{-1}) & 1 \end{bmatrix}\right) \begin{bmatrix} a_5 & 0 \\ 0 & a_5^{-1} \end{bmatrix}$

FB3: $\mathbf{E}(z) = \left(\prod_{i=1}^{3} \begin{bmatrix} 1 & 0 \\ a_{2i-1}\gamma(z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} 1 & a_{2i}\gamma(z) \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} a_7 & 0 \\ 0 & a_7^{-1} \end{bmatrix}$

FB4: $\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ a_1(z-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ a_2 & 1 \end{bmatrix} \left(\prod_{i=3}^{4} \begin{bmatrix} 1 & a_i \\ a_i z^{-1} & z^{-1} \end{bmatrix}\right)$

First three FBs are all type-B and are factorized in a similar manner while the last FB is of type-A and is factorized in a hybrid way. The last factorization consists of 3 lattice steps and 1 lifting step. These factorizations are all canonical in the sense that their number of free parameters is equal to the degree of freedom of that FB.

For FB1, compared to the previous factorization, the factorization here leads to a more efficient implementation. We now have only 3 parameters (without counting reciprocal) instead of 5 for direct form. For the other 3 FBs, the proposed factorizations also achieve the advantage of dramatic decrease in the number and dynamic range of the filter coefficients. Using the lifting structure, the number of coefficients reduces from 9 (direct form) to 5 for FB2, from 13 to 7 for FB3 and from 8 to 4 for FB4. Similarly, the dynamic range reduces from 36 to 17 for FB2 (for reference in [6], a similar factorization gives the dynamic range of 30), from 21 to 11 for FB3, from 221 to 20 for FB4. Therefore, the proposed factorization is typically capable of a more efficient implementation.

	h_0 (FB3)	h_1 (FB3)	h_0 (FB4)	h_1 (FB4)
0	0.767245	0.832848	0.788486	0.615051
1	0.383269	-0.448109	0.047699	-0.133389
2	-0.06888	-0.069163	-0.129078	-0.067237
3	-0.033475	0.109737		-0.006724
4	0.047282	0.006292		0.018914
5	0.003759	-0.014182		
6	-0.008473			

Table 2. The filter coefficients for the existing FBs

4.2. Reduced precision implementation

Due to the structurally PR as well as LP, the requirements stated above can be further reduced by representing the lifting or lattice coefficients with less precision while still retaining LP and PR properties. This is quite useful when the resource for implementation is limited or we prefer least cost. We demonstrate this with FB2, FB3 and FB4. We use from infinite bits (double precision

	FB2	FB3	FB4
a_1	0.58613434191	0.59742875528	0.14653358552
a_2	0.66806717120	1.54015786729	0.45537754123
a_3	-0.0700180094	0.11059029412	-9.6890747365
a_4	-1.2001710166	-0.1903980712	-2.7060935152
a_5	1.14960439886	-0.1972320984	
a_6		-2.1812627017	
a_7		1.01651054891	

Table 3. The coefficients of factorization

floating point) to 7 bits (sign bit included) to denote the coefficients. The frequency response for different bits of the last two FBs is depicted in Fig. 4 and 5. We find the benefits of reduced implementation requirements are only at the cost of marginal deterioration of the frequency response. Also, we compare the performance of these reduced precision FBs to that of infinite precision for image coding in Table 4. The results are the average difference of coding performance from the infinite bits case in terms of PSNR (dB) for several images with 5-level wavelet based zero-tree compression at the rate of 1bpp. As we can see from the table, the performance is acceptable for the precision with more than 7 bits (inclusive for FB4).

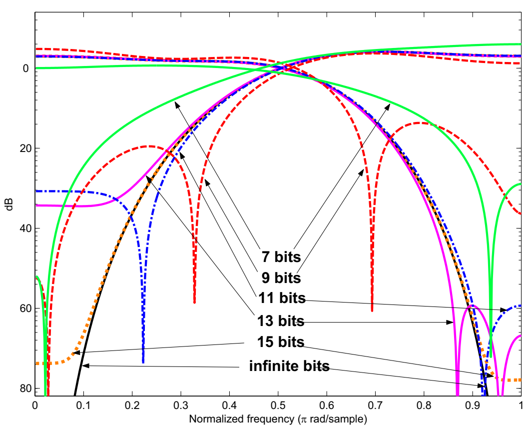

Fig. 4. The frequency response of FB3

	15 bits	13 bits	11 bits	9 bits	7 bits
FB2	0.0019	0.0058	0.0332	0.5289	0.9567
FB3	0.0009	0.2347	0.3411	0.6086	2.2579
FB4	0.0013	0.0102	0.0479	0.0908	0.1100

Table 4. Average PSNR change in dB for different precisions

Note that during the reduction of the precision, the vanishing moment (VM) of the FB is also one desirable property for applications such as image compression. Due to the nature of type-A FB, one VM will be enforced automatically. However this is not true for type-B. That's the reason for the different performance between these two types of FBs in Table 4.

In fact, the proposed factorization is used to retain the VM for some FBs. For example, for FB1, if the coefficients $-\frac{1}{2}$, $\frac{1}{4}$, $\sqrt{2}$ in reduced precision are denoted by a, b, c, VM for the low pass filter is retained when $4abc^2 + 2a + c^2 = 0$ is satisfied. This condition

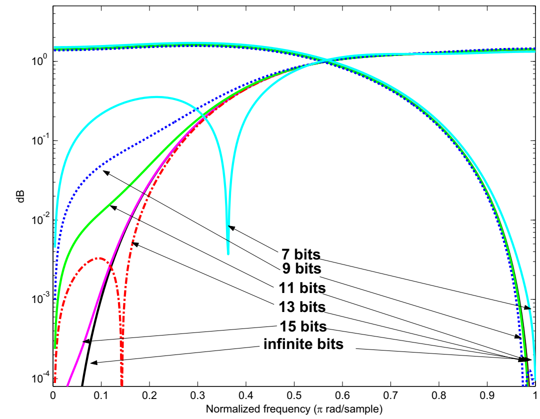

Fig. 5. The frequency response of FB4

is achieved by putting $z = -1$. In the above, the reduced precision coefficients to FB2 and FB3 are chosen in a similar fashion to retain the VM. However, for the reduced precision case, the adjustment of the coefficient for VM generally contradicts the aim of restraining the deterioration of frequency response. Hence, to find out the optimum tradeoff is interesting and worth further study.

5. CONCLUSION

In this paper, we study the relation between lifting structure and 2-channel LPPRFB. For type-B, an effective lifting structure is presented. We prove it to be complete by factorizing an arbitrary type-B FB. For type-A a lifting factorization is proposed which is equivalent to lattice factorization. For both of them the LPPR property can be imposed to the structure. We demonstrate the factorization by some examples. As an application, we demonstrate the use of the lifting structure for reduced precision FBs in image coding. It has been shown that the performance of the FBs are similar while their coefficients are implemented with less bits.

6. REFERENCES

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