MULTI-DIMENSIONAL AVERAGE-INTERPOLATING REFINEMENT ON ARBITRARY LATTICES

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ABSTRACT

Multi-dimensional datasets containing local averages of a function arise in many applications such as processing of CCD captures and medical images. Motivated by this fact we introduce multi-dimensional average-interpolating refinement on arbitrary lattices in arbitrary dimensions. Our refinement algorithm results in smooth scaling functions of compact support. This method forms a basis for multi-dimensional multi-resolution analysis and subdivision on datasets obtained by locally averaging a smooth function. As an example we present two-dimensional polynomial average-interpolating subdivision on the quincunx lattice and show that the resulting scaling functions are highly regular in the sense of Sobolev.

1. INTRODUCTION

Refinement algorithms play a central role in multi-resolution analysis and wavelet-based signal processing, e.g. in the lifting scheme [1], and also in a variety of other fields, such as computer graphics and numerical solutions to partial differential equations, where subdivision algorithms have proved to be useful. We refer the reader to [2] and [3] for general reviews of wavelets and stationary subdivision.

Until recently, most of the work in multidimensional wavelet-based signal processing relied on a separable extension of one-dimensional wavelets using tensor products. This approach, however, is restrictive and gives preference to certain directions. For this reason designing multi-resolution analysis tools for non-separable lattices in multiple dimensions has become important [4, 1]. These tools also allow wavelet-like analysis on manifolds [5].

In many applications one encounters datasets containing local averages of a function defined over a bounded domain in a multi-dimensional Euclidean space. Examples include 2D CCD captures and 3D CT scans. A refinement method that would in the limit provide a smooth function with the same local averages is thus desirable. Donoho in [6] introduced one-dimensional average-interpolation and resulting $\mathbf{L}^2(\mathbb{R})$ wavelets on a uniform one-dimensional grid. In this paper we extend this idea and develop average-interpolation on *arbitrary* lattices in *arbitrary* dimensions. Multivariate polynomial interpolation has been previously studied in the work of Kovačević and Sweldens [7]. The authors, however, know of no systematic method for "average"-interpolation on arbitrary lattices in arbitrary dimensions in the literature, except for the present exposition.

In what follows, after a review of lattices in $\mathbb{R}^m$, in section 2 partitionings of arbitrary domains are presented. Multi-dimensional average-interpolation on arbitrary lattices is developed in section 3. Section 4 contains as an example a useful special case: polynomial average-interpolation on the quincunx (a.k.a. checker-board, red-black) lattice. Finally, section 5 concludes the paper.

2. LATTICES AND PARTITIONINGS

In the following, $\mathbb{N}_0^m$ stands for the set of non-negative integer m-tuples, and $\mathbf{x}^\mathbf{p}$ where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{p} \in \mathbb{N}_0^m$ denotes $\prod_{1\le i\le m} x_i^{p_i}$. Let $|\mathbf{p}| := \sum_{1\le i\le m} p_i$. $\#(A)$ denotes the cardinality of the (finite) set A, and $\mu(R)$ represents a measure of the set R. We use terms from analysis and measure theory.

2.1. Lattices in $\mathbb{R}^m$

Lattices in $\mathbb{R}^m$ are formed from integral linear combinations of m linearly independent vectors:

$$L^{(0)} := G\mathbb{Z}^m,\ \det G \neq 0\ . \tag{1}$$

A non-singular integer matrix D with all eigenvalues strictly greater than 1 can be used to define $M = \det D$ (sub)lattices at level n:

$$L^{(n)} := GDG^{-1}L^{(n-1)} = GD^n\mathbb{Z}^m\ . \tag{2}$$

Since all of $\mathbb{Z}^m$ can be covered by $D\mathbb{Z}^m$ and $M-1$ translates of it, lattices satisfy the following refinement equation.

$$L^{(n-1)} = \bigcup_{0\le i<M} \left(L^{(n)} + t_i^{(n)}\right)\ . \tag{3}$$

Here $t_i^{(n)}$s are *sub-lattice identifiers* at level n and satisfy

$$t_i^{(n)} = GDG^{-1}t_i^{(n-1)} = GD^nG^{-1}t_i^{(0)} \ , \qquad (4)$$

where $t_i^{(0)}$s represent sub-lattices at level 0 and $t_0^{(n)} = \mathbf{0}$. If we restrict $G^{-1}t_i^{(0)}$s to the unit hypercube, they will be unique.

Equation (3) can be viewed as a two-scale relation that allows one to decompose a lattice at level $n-1$ into M sub-lattices at level n.

2.2. Partitionings

Definition 1. A *partitioning* $\Diamond_C^{\mathbf{R}}$ *of* $\mathbf{R} \subset \mathbb{R}^m$ *over* a countable set $C \subset \mathbb{R}^m$ is a collection of sets $\Diamond_C^{\mathbf{R}}[\zeta]$ of non-zero measure indexed by members of C,

$$\Diamond_C^{\mathbf{R}} = \{\Diamond_C^{\mathbf{R}}[\zeta] \subset \mathbf{R} | \zeta \in C\} \ , \qquad (5)$$

with the following properties:

1. $\bigcup_{\zeta\in C} \Diamond_C^{\mathbf{R}}[\zeta]$ is dense in $\mathbf{R}$,
2. $\Diamond_C^{\mathbf{R}}[\zeta_1] \cap \Diamond_C^{\mathbf{R}}[\zeta_2]$ is of zero measure if $\zeta_1 \neq \zeta_2$,
3. $\int_{\Diamond_C^{\mathbf{R}}[\zeta]} \mathbf{x}\,\mathrm{d}\mu = \zeta$, i.e. ζ is the centroid of $\Diamond_C^{\mathbf{R}}[\zeta]$.

We can construct a partitioning of $\mathbb{R}^m$ over $\mathbb{Z}^m$ by defining

$$\Diamond_{\mathbb{Z}^m}^{\mathbb{R}^m}[\mathbf{z}] := \{\mathbf{x} \in \mathbb{R}^m | z_i - \frac{1}{2} \leq x_i < z_i + \frac{1}{2},\ 1 \leq i \leq m\} \ . \qquad (6)$$

Families of partitionings over $L^{(n)}$s can be constructed using

$$\Diamond_{L^{(n)}}^{\mathbb{R}^m}[\zeta] := GD^n \Diamond_{\mathbb{Z}^m}^{\mathbb{R}^m}[\mathbf{z}] \qquad (7)$$

where $\mathbf{z} \in \mathbb{Z}^{\mathbf{m}}$ and $\zeta = GD^n\mathbf{z} \in L^{(n)}$.

Definition 2. A *restriction* of $\Diamond_C^{\mathbf{R}}$ to $E \subset C$ is denoted by $\Diamond_{C|E}^{\mathbf{R}}$ and defined as

$$\Diamond_{C|E}^{\mathbf{R}} := \{\Diamond_C^{\mathbf{R}}[\zeta] | \zeta \in E\} \ . \qquad (8)$$

It gives a partitioning of $\mathbf{R}_{C|E} := \text{clos} \bigcup_{\zeta\in E} \Diamond_C^{\mathbf{R}}[\zeta]$ on E.

Notation 1. We denote by $\mathbf{L}^1(\Diamond_C^{\mathbf{R}})$ the set of all functions that are μ-integrable over all $\Diamond_C^{\mathbf{R}}[\zeta]$s.

3. AVERAGE-INTERPOLATION

3.1. Average-Interpolation Operator

Definition 3. For a class of functions $\mathbf{F}$ and a set $E \subset L^{(n)}$, an operator

$$\mathfrak{A}_{L^{(n)}|E}^{\mathbf{F}} : \mathbf{L}^1(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}) \mapsto \text{clos Span } \mathbf{F} \qquad (9)$$

with the following property is a *class* $\mathbf{F}$ *average interpolation operator on* $\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}$.

$$\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta]} f\,\mathrm{d}\mu = \int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta]} \mathfrak{A}_{L^{(n)}|E}^{\mathbf{F}} f\,\mathrm{d}\mu, \qquad (10)$$
$$\forall \zeta \in E,\ f \in \mathbf{L}^1(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}) \ .$$

$\mathfrak{A}_{L^{(n)}|E}^{\mathbf{F}} f$ is said to be a *class* $\mathbf{F}$ *average-interpolation of* f *on* $\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}$. It is important to note that this average interpolation depends only on local averages of f on the partitioning (i.e. $\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta]} f\,\mathrm{d}\mu$'s) and not on the function itself.

Depending on the choice of $\mathbf{F}$ and E, this operator might not exist or might not be unique. However, the following theorem shows that we can always find such an operator for any finite $E \subset L^{(n)}$ and a class consisting of multivariate polynomials.

Theorem 1. *For any finite* $E \subset L^{(n)}$ *with* $\#(E) = K$, *one can always find a class* $\mathbf{\Pi}_{L^{(n)}|E}$ *average interpolation operator on* $\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}$, *where* $\mathbf{\Pi}_{L^{(n)}|E}$ *is a finite-dimensional class of multivariate polynomials.*

Proof. Denote members of E by ζ_i, $1 \leq i \leq K$, and let $\mathbf{p}_j$s give a sorting of $\mathbb{N}_0^m$. Consider the matrix M with elements

$$[M]_{ij} = \frac{\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_i]} \mathbf{x}^{\mathbf{p}_j}\,\mathrm{d}\mu}{\mu(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_i])} \ . \qquad (11)$$

Obviously $\text{rank} M_{K\times\infty} \leq K$, and it is not difficult to see that, due to the inclusion of all monomials of all degrees in the formation of M, equality always happens. Therefore, without loss of generality we can choose $\mathbf{p}_1, \ldots, \mathbf{p}_K \in \mathbb{N}_0^m$ with $|\mathbf{p}_1| \leq |\mathbf{p}_2| \leq \cdots \leq |\mathbf{p}_K|$ such that

$$P := \begin{pmatrix} \frac{\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_1]} \mathbf{x}^{\mathbf{p}_1}\,\mathrm{d}\mu}{\mu(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_1])} & \cdots & \frac{\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_1]} \mathbf{x}^{\mathbf{p}_K}\,\mathrm{d}\mu}{\mu(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_1])} \\ \vdots & \ddots & \vdots \\ \frac{\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_K]} \mathbf{x}^{\mathbf{p}_1}\,\mathrm{d}\mu}{\mu(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_K])} & \cdots & \frac{\int_{\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_K]} \mathbf{x}^{\mathbf{p}_K}\,\mathrm{d}\mu}{\mu(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m}[\zeta_K])} \end{pmatrix} \qquad (12)$$

is full-rank and $|\mathbf{p}_K|$ is minimum among all possible choices of such K m-tuples in $\mathbb{N}_0^m$. Let us denote by $\mathbf{\Pi}_{L^{(n)}|E}$ the class of multivariate polynomials produced by monomials $\mathbf{x}^{\mathbf{p}_1}, \ldots, \mathbf{x}^{\mathbf{p}_K}$.

For a sequence $a_E[\zeta]$, $\zeta \in E$, of local averages of an unknown function $f \in \mathbf{L}^1(\Diamond_{L^{(n)}|E}^{\mathbb{R}^m})$ we can therefore uniquely solve the system of equations

$$P \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \begin{pmatrix} a_E[\zeta_1] \\ \vdots \\ a_E[\zeta_K] \end{pmatrix} \ , \qquad (13)$$

which is equivalent to

$$\sum_{1\le j\le K} \alpha_j \frac{\int_{\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i]} \mathbf{x}^{\mathbf{p}_j}\,\mathrm{d}\mu}{\mu(\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i])} = a_E[\zeta_i] = \frac{\int_{\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i]} f\,\mathrm{d}\mu}{\mu(\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i])}\,, \tag{14}$$

$$\text{for } 1 \le i \le K\ .$$

Interchanging the order of summation and integration, for $1 \le i \le K$ we have

$$\int_{\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i]} \Big(\sum_{1\le j\le K} \alpha_j \mathbf{x}^{\mathbf{p}_j}\Big)\,\mathrm{d}\mu = \int_{\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}[\zeta_i]} f\,\mathrm{d}\mu\ . \tag{15}$$

Let $\mathfrak{A}^{\mathbf{\Pi}_{L^{(n)}|E}}_{L^{(n)}|E} f := \sum_{1\le j\le K} \alpha_j \mathbf{x}^{\mathbf{p}_j}$. We have therefore found a class $\mathbf{\Pi}_{L^{(n)}|E}$ average-interpolation of an arbitrary $\mathbf{L}^1(\Diamond^{\mathbb{R}^m}_{L^{(n)}|E})$ function f on $\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}$. □

E can be chosen such that $\mathbf{\Pi}_{L^{(n)}|E}$ average-interpolates $\Diamond^{\mathbb{R}^m}_{L^{(n)}|E}$ and its translates $\Diamond^{\mathbb{R}^m}_{L^{(n)}|E+t_i^{(n)}}$. Moreover, dilated versions of the monomials can be used at successive scales. We use this fact in section 4 to keep the refinement mask fixed across scales.

3.2. Average-Interpolation on $L^{(n)}$ Lattices

Using the average-interpolation operator introduced in the previous subsection, we define an average-interpolation refinement scheme that exactly reproduces multivariate polynomials up to a certain degree. This scheme consists of two steps:

1. Insertion: In this step new vertices are inserted in the lattice according to (3) and their associated values are set equal to the local averages on a finer partition, of a function that average-interpolates neighbouring samples.

2. Update: In this step the original sample values are updated to match the local averages of the average-interpolating function on the new (finer) partitioning.

Algorithm 1, that operates on a sequence $\{a^{(n)}[\zeta]\}$ defined on $L^{(n)}$ to create a sequence $\{a^{(n-1)}[\zeta]\}$ on a finer lattice $L^{(n-1)}$, demonstrates our scheme for $M = 2$. Here, $E^{(n)}$ denotes a pre-chosen average-interpolation neighbourhood.

From the proof of theorem 1 we have for $E' \subset L^{(n)}$ and $V \subset \mathbb{R}^m$

$$\begin{aligned}
&\int_V \frac{\mathfrak{A}^{\mathbf{\Pi}^{(n)}}_{L^{(n)}|E'} f}{\mu(V)}\,\mathrm{d}\mu \\
&\quad= \frac{\int_V \Big(\sum_{1\le j\le K} \alpha'_j \mathbf{x}^{\mathbf{p}_j}\Big)\,\mathrm{d}\mu}{\mu(V)} \\
&\quad= \sum_{1\le j\le K} \alpha_j \frac{\int_V \mathbf{x}^{\mathbf{p}_j}\,\mathrm{d}\mu}{\mu(V)} \\
&\quad= \begin{pmatrix} \frac{\int_V \mathbf{x}^{\mathbf{p}_1}\,\mathrm{d}\mu}{\mu(V)} & \cdots & \frac{\int_V \mathbf{x}^{\mathbf{p}_K}\,\mathrm{d}\mu}{\mu(V)} \end{pmatrix} \begin{pmatrix} \alpha'_1 & \cdots & \alpha'_K \end{pmatrix}^T \\
&\quad= \begin{pmatrix} \frac{\int_V \mathbf{x}^{\mathbf{p}_1}\,\mathrm{d}\mu}{\mu(V)} & \cdots & \frac{\int_V \mathbf{x}^{\mathbf{p}_K}\,\mathrm{d}\mu}{\mu(V)} \end{pmatrix} P^{-1} \begin{pmatrix} a_{E'}[\zeta_1] \\ \vdots \\ a_{E'}[\zeta_K] \end{pmatrix}.
\end{aligned} \tag{16}$$

Insertion and update steps in algorithm 1 can therefore be implemented using refinement masks.

By setting $E^{(n)} = GD^nG^{-1}E^{(0)}$, this scheme can be iterated, resulting in an average-interpolating function in the limit, as for any integrable function g

$$\lim_{n\to-\infty} \frac{\int_{\Diamond^{\mathbb{R}^m}_{L^{(n)}}[\zeta']} g\,\mathrm{d}\mu}{\mu(\Diamond^{\mathbb{R}^m}_{L^{(n)}}[\zeta'])} = g(\zeta')\ . \tag{17}$$

Depending on the choice of the average-interpolation neighbourhoods $E^{(n)}$, this scheme can reproduce certain polynomials up to a degree $N \le |\mathbf{p}_K|$. More specifically, if E is chosen such that $\mathbf{\Pi}^{(n)}_i$s and $\mathbf{\Pi}^{(n)}_u$s are translation-invariant, all polynomials in $\bigcap_n(\mathbf{\Pi}^{(n)}_i \cap \mathbf{\Pi}^{(n)}_u)$ can be exactly reproduced.

4. EXAMPLE: QUINCUNX AVERAGE-INTERPOLATING SUBDIVISION

For quincunx average-interpolation, one possible choice for D and G matrices is

$$D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\ G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{18}$$

which corresponds to a 45° rotation and $\sqrt{2}$ scaling at each level.

We use neighbourhood rings for average interpolation, as depicted in figure 1, and choose our average interpolation operator such that insertion and update masks do not change through scales. This is achieved by using a 45° rotated version of the monomials at every other scale.

Insertion and update masks for closest neighbourhoods of sizes 4 and 12 are summarized in table 1. These masks exactly reconstruct polynomials of up to degree 1 and up to degree 3 respectively. The associated scaling functions, shown in figure 2, are of compact support. Moreover, the Sobolev regularity exponents for these scaling functions, calculated using the algorithm of [8], are significantly higher than the values reported in [7] for quincunx interpolating scaling functions of the same order (see table 2).

Algorithm 1 Average-interpolation refinement scheme

1. Insertion:
   for $\zeta \in L^{(n)}$

$$a^{(n-1)}[\zeta + t^{(n)}] = \frac{\int_{\Diamond^{\mathbb{R}^m}_{L^{(n-1)}}[\zeta+t^{(n)}]} \mathfrak{A}^{\mathbf{\Pi}_i^{(n)}}_{L^{(n)}|(E^{(n)}+\zeta+t^{(n)})} f \, \mathrm{d}\mu}{\mu(\Diamond^{\mathbb{R}^m}_{L^{(n-1)}}[\zeta + t^{(n)}])}$$

   end for.

2. Update:
   for $\zeta \in L^{(n)}$

$$a^{(n-1)}[\zeta] = \frac{\int_{\Diamond^{\mathbb{R}^m}_{L^{(n-1)}}[\zeta]} \mathfrak{A}^{\mathbf{\Pi}_u^{(n)}}_{L^{(n-1)}|(E^{(n)}+\zeta)} f \, \mathrm{d}\mu}{\mu(\Diamond^{\mathbb{R}^m}_{L^{(n-1)}}[\zeta])}$$

   end for.

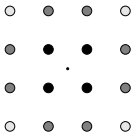
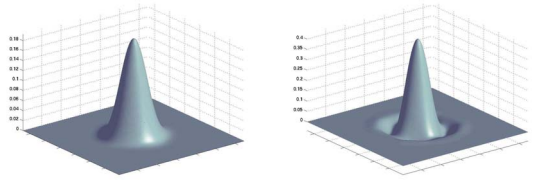


Fig. 1. Three neighbourhood rings, shown in shades of gray. The dot marks the position of the new (inserted) sample.



(a) for neighbourhood of size 4. (b) for neighbourhood of size 12.

Fig. 2. Quincunx average interpolating scaling functions.

Table 1. Quincunx insertion and update masks.

Subdivision neighbourhood size	Operation	Mask coefficients on ring 1	on ring 2
4	insertion	0.2500	–
4	update	0.2500	–
12	insertion	0.3229	−0.0365
12	update	0.3125	−0.0313

Table 2. Comparison of lower bounds on Sobolev exponents of average-interpolating (AI) and interpolating (I) scaling functions.

Order	Sobolev exp. (AI)	Sobolev exp. (I)
1	4.859291	1.577645
3	5.070267	2.447923

5. CONCLUSION

In this work we have introduced average-interpolating refinement on arbitrary lattices in arbitrary dimensions. This kind of refinement is especially useful for applications where data samples are obtained by locally averaging a smooth underlying function. As an example we have presented quincunx average-interpolating subdivision and have shown that the resulting scaling functions are significantly more regular than interpolating scaling functions introduced in [7]. In an upcoming paper we will introduce multi-dimensional average-interpolating wavelets on arbitrary grids.

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7. REFERENCES

[1] W. Sweldens and P. Schröder, "Building your own wavelets at home," in *Wavelets in Computer Graphics*, pp. 15–87. ACM SIGGRAPH Course notes, 1996.

[2] Stéphane Mallat, *A wavelet tour of signal processing*, Academic Press, 2nd edition, 1999.

[3] Alfred S. Cavaretta, Wolfgang Dahmen, and Charles A. Micchelli, "Stationary subdivision," *Mem. Amer. Math. Soc.*, vol. 93, no. 453, September 1991.

[4] W. Sweldens, "The lifting scheme: A construction of second generation wavelets," *SIAM J. Math. Anal.*, vol. 20, no. 2, pp. 511–546, March 1998.

[5] P. Schröder and W. Sweldens, "Spherical wavelets: Efficiently representing functions on the sphere," *Computer Graphics Proceedings (SIGGRAPH 95)*, pp. 161–172, 1995.

[6] David L. Donoho, "Smooth wavelet decomposition with blocky coefficient kernels," in *Recent advances in wavelet analysis*, Larry L. Schumaker and Glenn Webb, Eds., pp. 258–308. Academic Press, 1994.

[7] J. Kovačević and W. Sweldens, "Wavelet families of increasing order in arbitrary dimensions," *IEEE Trans. Image Proc.*, vol. 9, no. 3, pp. 480–496, March 2000.

[8] Amos Ron, Zuowei Shen, and Kim-Chuan Toh, "Computing the Sobolev regularity of refinable functions by the Arnoldi method," *J. Matrix Anal. Appl.*, vol. 23, no. 1, pp. 57–76, 2001.