MULTIRESOLUTION OF THE FOURIER TRANSFORM

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ABSTRACT

In this paper, the integral Fourier transform is described as a specific wavelet-like transform with a fully scalable modulated window, but not all possible translations. The transform is defined by sinusoidal waves of half periods. A geometrical locus of frequency-time points for the proposed wavelet-like transform is derived and the function coverage is described and compared with the short-time Fourier transform as well as with the wavelet transform.

1. INTRODUCTION

The Fourier transform is considered traditionally as a transform without time resolution, since the basis cosine and sine functions are defined everywhere on the real line. Each Fourier component depends on the global behavior of the signal. Wavelet analysis has been developed as multiresolution signal processing, which is used effectively for signal and image processing, compression, computer vision, medical imaging, etc. [1]-[3]. In wavelet analysis, a fully scalable modulated window is used for frequency localization [4]-[6]. The window is sliding, and the wavelet transform of a part of the signal is calculated for every position. The result of the wavelet transform is a collection of time-scaling representations of the signal with different resolutions.

In this paper, a new representation of the Fourier transform is described, by cosine and sine wavelet-like transforms with fully scalable modulated windows. The integral Fourier transform uses a specific collection of time-scaling representations of the signal with different resolutions. The transform can be also considered as a discretization of a integral wavelet transform. The Fourier transform uses the translations of windows not for every position. The transform provides the multiresolution signal processing because cosine and sine type waveforms of every frequency are participated in Fourier analysis.

2. FOURIER TRANSFORM WAVELET

We describe the integral Fourier transform by wavelet-like transforms with cosine and sine analyzing functions. The description differs from the well-known concept of the short-time Fourier transform or windowed Fourier transform [1, 7, 8]. This transform is based on a joint time-frequency signal representation and is defined by

$$F(t,\omega)=\int_{-\infty}^{\infty} f(\tau)g(\tau-t)e^{-j\omega\tau}d\tau \qquad (1)$$

where $t,\omega \in (-\infty,+\infty)$. A time-sliding window function $g(t)$ is used to emphasize "local" frequency properties. The window function is typically considered to be symmetric and with unit norm in the space of square-integrable functions. For instance, the Gaussian function $g(t)=(\sqrt{\pi\sigma})^{-1}$ $\exp(-t^2/\sigma)$ with a symmetric finite support can be taken, where $\sigma > 0$ is a fixed number defining a "width" of the window.

The inverse short-time Fourier transform is defined by

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau,\omega)g(t-\tau)e^{j\omega t}d\omega d\tau. \qquad (2)$$

Let $\psi(t)$ and $\varphi(t)$ be functions that coincide respectively with the cosine and sine functions inside the half of period interval $[-\pi/2,\pi/2)$ and equal zero outside this interval

$$\begin{aligned}\psi(t) &= \begin{cases}\cos(t), & t\in[-\pi/2,\pi/2)\\ 0, & \text{otherwise}\end{cases}\\ \varphi(t) &= \begin{cases}\sin(t), & t\in[-\pi/2,\pi/2)\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \qquad (3)$$

We consider a family $\{\psi_{\omega;b_n}(t),\varphi_{\omega;b_n}(t)\}$ of time-scale and shift transformations of these functions

$$\begin{aligned}\psi_{\omega;b_n}(t) &= \psi(\omega[t-b_n])\\ \varphi_{\omega;b_n}(t) &= \varphi(\omega[t-b_n])\end{aligned}$$

where $t\in(-\infty,+\infty)$, frequency ω varies along the real line, and b_n takes values of a finite or infinite set to be defined below. Let $f(t)$ be a function for which the Fourier transform exists

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt.$$

For a given frequency $\omega \neq 0$, the process of $F(\omega)$ component formation, when the cosine and sine waveforms of frequency ω are interfering with $f(t)$, can be described in the following way. We consider the partition of the time line by intervals of length π/ω with centers at integer multiples of π/ω

$$I_n = I_n(\omega) = \left[\frac{(2n-1)\pi}{2\omega}, \frac{(2n+1)\pi}{2\omega}\right), \; n = 0, \pm 1, \pm 2, \dots .$$

The integral of the Fourier transform $F(\omega)$, $\omega \neq 0$, can be split by intervals $I_n(\omega)$ and written as follows

$$\begin{aligned} F(\omega) &= \sum_{n=-\infty}^{\infty} \int_{I_n} f(t)\cos(\omega t)dt - j\sum_{n=-\infty}^{\infty} \int_{I_n} f(t)\sin(\omega t)dt \\ &= \sum_{n=-\infty}^{\infty} (-1)^n \left[\int_{-\infty}^{\infty} f(t)\psi\left(\omega\left[t - \frac{\pi}{\omega}n\right]\right)dt\right. \\ &\quad \left. -j\int_{-\infty}^{\infty} f(t)\varphi\left(\omega\left[t - \frac{\pi}{\omega}n\right]\right)dt\right]. \end{aligned}$$

We now introduce the following transforms of the function $f(t)$

$$\begin{aligned} F_\psi(\omega, b_n) &= \int_{-\infty}^{\infty} f(t)\psi_{\omega,b_n}(t)\,dt \\ F_\psi(0,0) &= \int_{-\infty}^{\infty} f(t)dt \end{aligned} \tag{4}$$

and

$$\begin{aligned} F_\varphi(\omega, b_n) &= \int_{-\infty}^{\infty} f(t)\varphi_{\omega,b_n}(t)\,dt \\ F_\psi(0,0) &= 0 \end{aligned} \tag{5}$$

where $b_n = b_n(\omega) = (\pi/\omega)n$, when n is an integer. It is assumed that $b_n \equiv 0$ if $\omega = 0$.

$F_\psi(\omega, b_n)$ is the integral of the cosinusoidal signal $\psi(\omega t)$ of the half-period π/ω, which is multiplied on $f(t)$ waveform at location b_n. The signal $\psi(\omega t)$ is moving along $f(t)$ waveform at locations b_n being integer multiples of π/ω. The locations depend on the frequency ω. In a similar way, the transform $F_\varphi(\omega, b_n)$ is defined by the inner product of $f(t)$ waveform with the sinusoidal signal $\varphi(\omega t)$ of the half-period π/ω, when moving at the same locations b_n.

The complex Fourier transform is composed by the pair of real transforms $F_\psi(\omega, b_n)$ and $F_\varphi(\omega, b_n)$ as

$$F(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n F_\psi(\omega, b_n) - j\sum_{n=-\infty}^{\infty} (-1)^n F_\varphi(\omega, b_n). \tag{6}$$

These transforms are calculated in the following set of points in the frequency-time plane

$$\begin{aligned} B = \Big\{ &(\omega, b_n);\; \omega \in (-\infty, +\infty),\; b_n = n\frac{\pi}{\omega}, \\ &n = 0, \pm 1, \pm 2, \dots, \pm N(\omega) \Big\} \end{aligned} \tag{7}$$

where $N(0) = 0$, and $N(\omega)$ is the infinite in general, or a finite number if the function $f(t)$ has a finite support. Given an integer n, we call the set of points $B_n = \{(\omega, b_n);\; \omega \in (-\infty, +\infty)\}$ to be the nth center-line in the frequency-time plane. The locus B is the union of such center-lines, $B = \{\cup B_n;\; n = 0, \pm 1, \dots, \pm N(\omega)\}$.

Let $f(t)$ be $\cos(\omega_0 t)$ waveform with frequency $\omega_0 = 1.3$, which is defined in the interval $[-3\pi, 3\pi]$. We consider the Fourier transform of the waveform in the frequency interval $[0, 3\pi]$. Figure 1 shows the following set of frequency-time points

$$\begin{aligned} B = \Big\{ &(\omega, b_n);\; \omega \in (0, 3\pi),\; b_n = \frac{\pi n}{\omega}, \\ &n = 0, \pm 1, \pm 2, \dots, \pm N(\omega) \Big\} \end{aligned} \tag{8}$$

where $N(\omega)$ is defined as

$$N(\omega) = \begin{cases} 0, & \text{if } \omega < 1/6 \\ \left\lfloor 3\omega - \frac{1}{2} \right\rfloor, & \text{if } \omega \in (1/6, 3\pi) \end{cases} \tag{9}$$

and the operation $\lfloor \cdot \rfloor$ denotes the floor function. The number of center-lines in the set B is 55, i.e. $N(\omega) \leq 27$. Horizontal lines with centers located at a few points (ω, b_n) of the set B show the widths, π/ω, of the corresponding cosine and sine basis signals $\varphi(\omega t)$ and $\psi(\omega t)$ of transforms $F_\psi(\omega, b_n)$ and $F_\varphi(\omega, b_n)$, respectively.

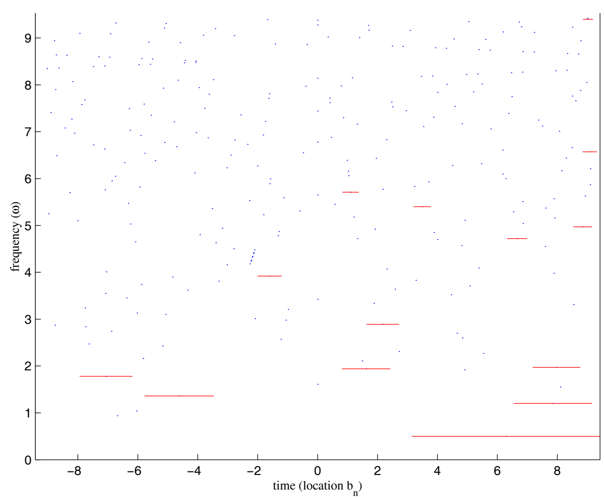

Fig. 1. The locus of time-frequency points for the B-wavelet transform.

The function $N(\omega)$ defines the required number of shifted versions of signals $\psi(\omega t)$ and $\varphi(\omega t)$ for calculation of values of transforms $F_\psi(\omega, b_n)$ and $F_\varphi(\omega, b_n)$ for frequency ω. For instance, when $0 < \omega < 1/6$ the only signals $\psi(\omega t)$ and $\varphi(\omega t)$ will be multiplied with $f(t)$ and then integrated to define $F_\psi(\omega, 0)$ and $F_\varphi(\omega, 0)$. No more translations of these signals are required for calculation of the Fourier transform $F(\omega)$ at these frequencies. We can consider that $F_\psi(\omega, b_n) = 0$ and $F_\varphi(\omega, b_n) = 0$ if $n \neq 0$.

When $\omega \in (1/6, 1/2)$ the signals $\psi(\omega t)$ and $\varphi(\omega t)$ are required to be shifted to the right and left by π/ω. To calculate the Fourier transform $F(\omega)$ at such frequencies, the only following values are needed

$$F_\psi(\omega, 0)\,,\; F_\psi\left(\omega, \frac{\pi}{\omega}\right), F_\psi\left(\omega, -\frac{\pi}{\omega}\right)$$

and

$$F_\varphi(\omega, 0)\,,\; F_\varphi\left(\omega, \frac{\pi}{\omega}\right), F_\varphi\left(\omega, -\frac{\pi}{\omega}\right).$$

In the general case, the number of required translations $M(\omega) = 2N(\omega) + 1$ can be defined as the following step function

$$M(\omega) = \begin{cases} 1, & \text{if } 0 < |\omega| \le \dfrac{1}{6} \\ 2n+1, & \text{if } \dfrac{2n+1}{6} < |\omega| \le \dfrac{2n+3}{6}. \end{cases}$$

We can consider that $F_\psi(\omega, b_n) = 0$ and $F_\varphi(\omega, b_n) = 0$, when $|n| > N(\omega)$.

To calculate the Fourier transform of $f(t)$, the values of transforms $F_\psi(\omega, b_n)$ and $F_\varphi(\omega, b_n)$ are required at frequency-time points of set B. Figure 2 shows the 3-D plot of required values of the transform $\{F_\psi(\omega, b_n);\ (\omega, b_n) \in B\}$ versus the frequency ω and location b. The 3-D plot of the transform $\{F_\varphi(\omega, b_n);\, (\omega, b_n) \in B\}$ is given in Fig. 3.

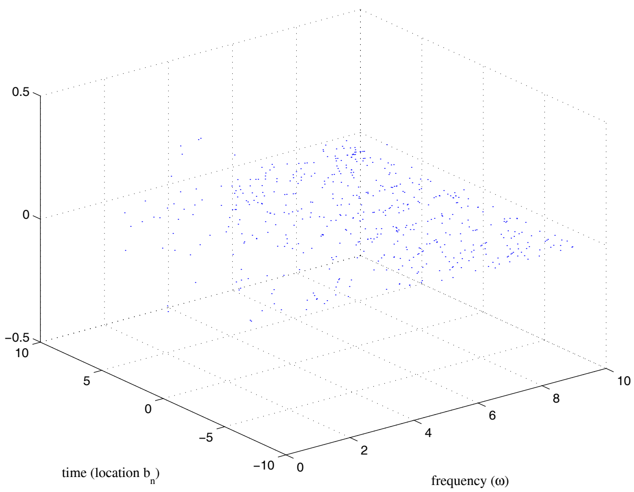

Fig. 2. Wavelet transform plot of the signal $f(t) = \cos(\omega_0 t)$ with the cosine analyzing function.

The representation of $f(t)$ by the pair of transforms $F_\psi(\omega, b_n)$ and $F_\varphi(\omega, b_n)$ (or by $F_\psi(\omega, b_n) - jF_\varphi(\omega, b_n)$) we name to be the B-wavelet transform. The transforms

$$f(t) \to \{F_\psi(\omega, b_n), (\omega, b_n) \in B\}$$
$$f(t) \to \{F_\varphi(\omega, b_n), (\omega, b_n) \in B\}$$

we call to be respectively the cosine and sine B-wavelet transforms.

Set B can be considered as an "optimal" geometrical locus of frequency-time points for the integral Fourier transform defined by the B-wavelet transform. There is no need in calculation of the B-wavelet transform across the whole range of frequency-time points, but the points of B set. The integral Fourier transform can be derived from B-wavelet transform by (6).

The analysis of the introduced B-wavelet representation shows that the Fourier transform can be described as a pair of integral wavelet transforms sampled by only translation

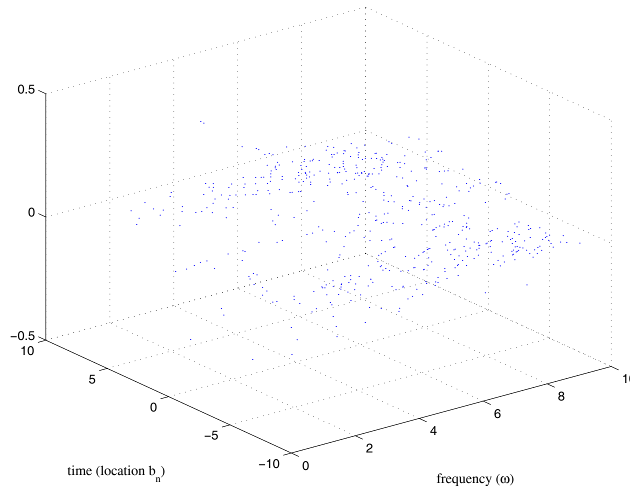

Fig. 3. Wavelet transform plot of the signal $f(t) = \cos(\omega_0 t)$ with the sine analyzing function.

parameter. These wavelet transforms are defined by

$$\begin{aligned} T_\psi(\omega, b) &= \int_{-\infty}^{\infty} f(t)\psi_{\omega,b}(t)dt \\ T_\varphi(\omega, b) &= \int_{-\infty}^{\infty} f(t)\varphi_{\omega,b}(t)dt \end{aligned} \tag{10}$$

where $\omega, b \in (-\infty, \infty)$. The integral of $\psi(t)$ does not equal zero. However the simple shift by $\pi/2$ yields this property, and such a shift can be taken as a basis function for the transform T_ψ. In this case, the locus for this transform will be another set of frequency-time points $B' = \{(\omega, b'_n);\, b'_n = b_n + \pi/2\omega,\ \omega \in (-\infty, +\infty),\ n = 0, \pm 1, \pm 2, \ldots\}$.

Locus B of frequency-time points differs from grids used for sampling the short-time Fourier transform and the wavelet transform. For the short-time Fourier transform, a single window is used for all frequencies. The resolution of the analysis is the same at all locations in the time-frequency domain. When sampling the short-time Fourier transform,

$$F(t, \omega) \to F_{n,m} = F(nt_0, m\omega_0),\ n, m = 0, \pm 1, \ldots,$$

a regular rectangular grid is used with time and frequency steps t_0 and ω_0, that satisfy the frame bound condition $t_0\omega_0 \le 2\pi$ [9]. One can note that for the B-wavelet transform the condition $\omega b_1(\omega) = \pi$ holds for all frequencies ω.

For traditional integral wavelet transforms, short high-frequency and long low-frequency windowed functions are used for all translations. The windows are overlapped, because of continuously shifting them, and the wavelet coefficients are therefore highly redundant. When sampling the wavelet transform $T(a, b)$, two parameters are chosen, a time-step $a_0 > 1$ and location $b_0 > 0$. The frames are constructed by sampling the dilation exponentially $a = a_0^n$ and the translation b proportionally a_0^n, as follows

$$T(a, b) \to T_{n,m} = T\left(a_0^n, mb_0 a_0^n\right),\ n, m = 0, \pm 1, \ldots.$$

We now illustrate changes in the B-wavelet representation of the Fourier transform when a signal is degraded by a short-time sinusoidal signal. Let $f(t)$ be $\cos(\omega_1 t)$ waveform of frequency $\omega_1 = \pi/3$ defined in the time interval $(-3\pi, 3\pi)$. We assume that in the interval $(-1.5, 1.5)$ a sinusoidal component $n(t) = 0.5\sin(\omega_2 t)$ has been added to $f(t)$. This short-time signal has a frequency six times that ω_1, i.e. $\omega_2 = 2\pi$, The signal to be analyzed by the B-wavelet-transform is (see Fig. 4)

$$g(t) = \begin{cases} f(t) + n(t), & |t| < 1.5 \\ f(t), & 1.5 \leq |t| < 3\pi. \end{cases} \tag{11}$$

Figure 5 shows the projection of the 3-D transform plot of

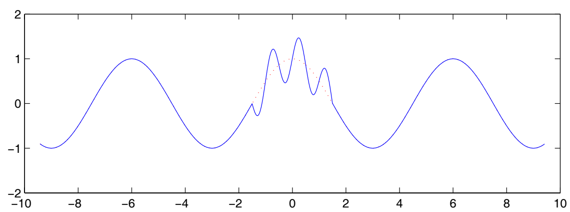

Fig. 4. Cosine waveform with a high-frequency short-time signal.

the original signal $f(t)$ in the time domain. T_ψ transform part of the B-wavelet transform is only illustrated. Figure 6 shows the projection of the 3-D B-wavelet transform plot of the observed signal $g(t)$ in the time domain. The addition noise signal $n(t)$ leads to the change of the transform plot at locations b_n lying in the interval $(-1.5, 1.5)$. Namely, main changes occur in the time interval $(-1.5, 1.5)$ along two center-lines $B_n = \{b_n(\omega) + \pi/\omega; \omega \in (0, 3\pi)\}$, where $n = 1$ and -1.

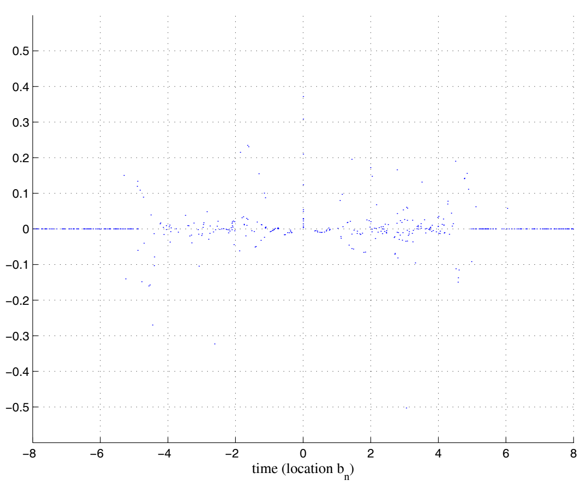

Fig. 5. Wavelet transform plot of the signal $f(t)$ with the cosine analyzing function (projection on the time domain).

3. CONCLUSION

A concept of the B-wavelet transform has been introduced and representation of the integral Fourier transform by this

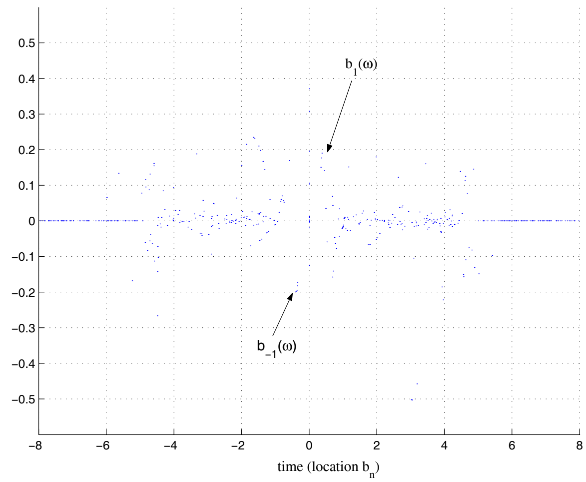

Fig. 6. Wavelet transform plot of the signal $g(t)$ with the cosine analyzing function (projection on the time domain).

transform has been described. The B-wavelet transform is defined on a specific set B of points in the frequency-time plane. This transform uses a fully scalable modulated window but not all possible locations. We assume to study and develop the Fourier approach to signal analysis and examine practical applications of the proposed B-wavelet transform in signal processing.

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