THE FINER DIRECTIONAL WAVELET TRANSFORM

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ABSTRACT

Directional information is an important and unique feature of multidimensional signals. As a result of a separable extension from one-dimensional (1-D) bases, the multidimensional wavelet transform has very limited directionality. Furthermore, different directions are mixed in certain wavelet subbands. In this paper, we propose a new transform that fixes this frequency mixing problem by using a simple "add-on" to the wavelet transform. In the 2-D case, it provides one lowpass subband and six directional highpass subbands at each scale. Just like the wavelet transform, the proposed transform is nonredundant, and can be easily extended to higher dimensions. Though nonseparable in essence, the proposed transform has an efficient implementation based on 1-D operations only.

1. INTRODUCTION

Directional information is a unique feature of multidimensional signals. Recently, the importance of directional information has been recognized by many image processing applications, including feature extraction, enhancement, denoising, classification, and compression.

The wavelet transform has a long and successful history as an efficient image processing tool. However, as a result of a separable extension from one-dimensional (1-D) bases, wavelets in higher dimensions can only capture very limited directional information. For instance, 2-D wavelets only provide three directional components, namely horizontal, vertical, and diagonal. Furthermore, the $45°$ and $135°$ directions are mixed in diagonal subbands.

There have been a number of systems, including the directional filter bank [1] and the complex wavelet transform [2, 3], that provide finer directional decomposition. However, the wavelet transform is still very attractive for image processing. In particular, it is nonredundant, and uses only 1-D operations. The complex wavelet transform is 4-times redundant for images, and in general 2^n-times redundant for n-dimensional signals. In terms of implementation, the wavelet transform can be implemented efficiently in a separable fashion. In contrast, systems such as the directional filter bank involve nonseparable filtering and sampling and have high computational complexity. Last but not least, the theory and applications of wavelets have already been extensively studied, offering us a plethora of ready-to-use filters and toolboxes.

Now the natural question is: Can we equip the wavelet transform with finer directionality, and still retain its desirable features? We give an affirmative answer in this paper by proposing a new finer directional wavelet transform. It can be seen as a simple "add-on" to the original wavelet system, and possesses the following properties. In the 2-D case, it produces one lowpass subband and six directional highpass subbands at each scale, and fixes the frequency mixing problem of wavelets. Like the wavelets, it is a nonredundant system, and can be easily extended to the higher dimensional case. Finally, the proposed transform has an efficient implementation based on 1-D operations only.

The outline of the paper is as follows. Section 2 presents the filter bank construction of the proposed system. Section 3 discusses filter design and efficient implementation. We will present some numerical results in Section 4 and conclude the paper in Section 5.

2. FILTER BANK CONSTRUCTION

We will first consider the 2-D case. The traditional way to construct 2-D wavelets is to use tensor products of their 1-D counterparts. The advantage of this approach is its simple separable implementation. Unfortunately, this also imposes serious limits on the directionality of the resulting frequency partitioning. As shown in Fig.1(a), the 2-D wavelet transform produces one lowpass subband (LL), and three highpass subbands (HL, LH, HH), corresponding to the horizontal, vertical, and diagonal directions. Furthermore, diagonal subbands mixes the directional information oriented at $45°$ and $135°$. The main idea of the proposed

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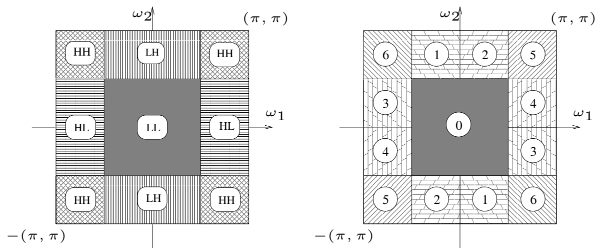
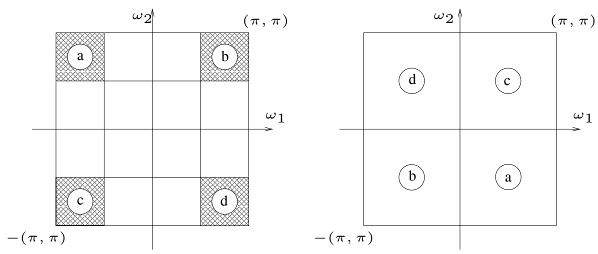


(a) (b)

Fig. 1. Division of the 2-D frequency spectrum. (a) The separable wavelet transform. (b) The proposed finer directional wavelet transform.



(a) (b)

Fig. 2. (a) The diagonal highpass frequency regions of the input signal. (b) The frequency contents of the diagonal subband (HH).

transform is to find a way to further divide each highpass region of the wavelets into two branches. In particular, we want to have a system with the frequency partitioning shown in Fig.1(b), which contains six directional subbands roughly oriented at $15°, 45°, 75°, 105°, 135°$ and $165°$. This is the same frequency decomposition provided by the 2-D complex wavelet transform [2], which has been shown to be successful in several image processing applications.

To construct the desired system, we first examine the frequency contents in the wavelet subbands. For example, we know the diagonal subband (HH) captures certain directional highpass frequency information (illustrated as regions a,b,c,d in Fig.2(a)) in the input signal, where a,d and b,c correspond to directional information oriented at $45°$ and $135°$ respectively. With the decimation operations in the wavelet transform, these frequency regions will be scrambled and mapped to the actual frequency contents in the HH subband, as shown in Fig. 2(b). Now to separate regions a,b from c,d, we can see that a natural choice is to use a two-channel filter bank with a checkerboard-shaped passband support, illustrated in Fig.3. The decimation matrix $\boldsymbol{D}_0$ in the filter bank is the simple diagonal matrix $diag(2, 1)$.

Actually, this same filter bank can also be used to divide the other two wavelets subbands (HL and LH). Now we can get a structured construction of the new system, as shown in Fig.4. Note that only the analysis part is given in the figure. The original wavelet transform is kept in the first two levels. In the third level, the three highpass subbands are further split by the checkerboard filter bank. To verify that the system really achieves the desired frequency partitioning, we can apply the noble identity [4] in multirate signal processing to get the equivalent filters of each subbands of the system (Fig. 5). Clearly the passbands of the equivalent filters exactly have the desired frequency supports.

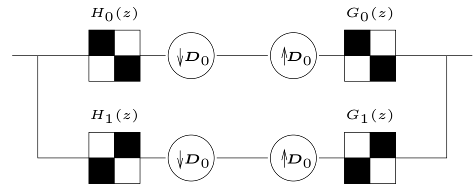


Fig. 3. The 2-D filter bank with a checkerboard-shaped passband support.

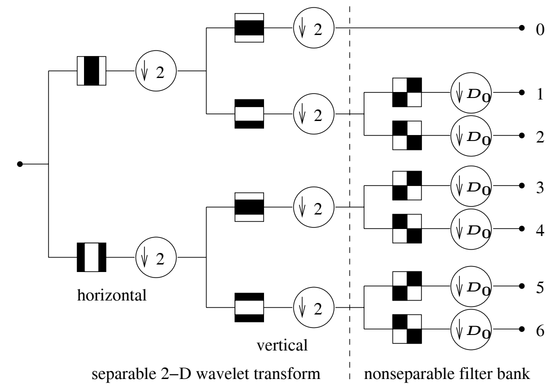


Fig. 4. The filter bank construction of the proposed system (analysis part).

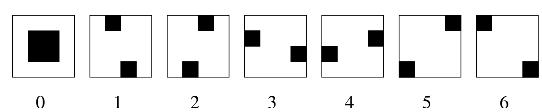


Fig. 5. The equivalent filters of subbands 0 - 6 of the proposed system. Dark regions represent passband.

Since each individual component of the proposed system, i.e. the wavelet transform and the checkerboard filter bank, is critically sampled, the overall system is also critically sampled. Furthermore, if we design the checkerboard filter bank to be perfect reconstruction, then the whole system is also perfect reconstruction.

Though we give the filter bank construction for the 2-D case, actually it can be generalized to arbitrary dimensions. The basic building blocks are still the 1-D wavelet decomposition, and 2-D checkerboard filter bank. The n-D system is still critically sampled, and remains prefect reconstruction. We will give details of this in a forthcoming paper.

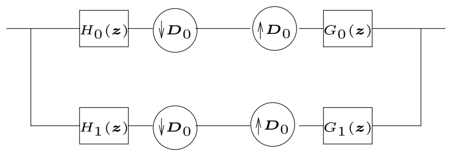


Fig. 6. 2-D filter bank with the decimation matrix $\boldsymbol{D}_0$.

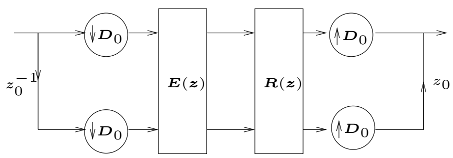


Fig. 7. The equivalent polyphase form of the filter bank.

3. FILTER DESIGN AND IMPLEMENTATION

For the first two stages of the proposed system, we can choose any existing wavelet filters to suit different applications. The remaining task is to design a 2-D checkerboard filter bank with the desired frequency response and perfect reconstruction. Here we will present a design based on a parametrization of the polyphase matrices. This form of parametrization was first proposed by Phoong etc. [5] for the 1-D filter bank. However, the 1-D to 2-D mapping method proposed in that work can only be used to design filters banks with a single parallelogram-shaped support, such as the diamond shape. While the checkerboard shape we want here does not belong to this class. In the following, we will propose a novel 1-D to 2-D mapping for the checkerboard filter bank in the polyphase domain.

Let us consider a two-channel maximally decimated filter bank in 2-D with the decimation matrix $\boldsymbol{D}_0$, shown in Fig. 6. The well-known equivalent polyphase form of the system is given in Fig. 7. We denote $\boldsymbol{z} = (z_0, z_1)$. The relations between the analysis and synthesis filters $\{H_k(\boldsymbol{z}), G_k(\boldsymbol{z})\}$ and the polyphase matrices $\boldsymbol{E}(\boldsymbol{z})$ and $\boldsymbol{R}(\boldsymbol{z})$ can be expressed as [4]

$$H_k(\boldsymbol{z}) = \boldsymbol{E}_{k,0}(z_0^2, z_1) + z_0^{-1}\boldsymbol{E}_{k,1}(z_0^2, z_1) \quad (1)$$

$$G_k(\boldsymbol{z}) = \boldsymbol{R}_{0,k}(z_0^2, z_1) + z_0\boldsymbol{R}_{1,k}(z_0^2, z_1), \quad (2)$$

for $k = 0, 1$. We can see from Fig.7 that the filter bank has perfect reconstruction if and only if

$$\boldsymbol{E}(\boldsymbol{z}) \cdot \boldsymbol{R}(\boldsymbol{z}) = \boldsymbol{I}, \quad (3)$$

where $\boldsymbol{I}$ is the identity matrix. In our design, we choose $\boldsymbol{E}(\boldsymbol{z})$ and $\boldsymbol{R}(\boldsymbol{z})$ to be

$$\boldsymbol{E}(\boldsymbol{z}) = \sqrt{2}\begin{pmatrix} 0.5 & 0 \\ -0.5\alpha(\boldsymbol{z}) & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha(\boldsymbol{z}) \\ 0 & z_0 \end{pmatrix} \quad (4)$$

and

$$\boldsymbol{R}(\boldsymbol{z}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -z_0^{-1}\alpha(\boldsymbol{z}) \\ 0 & z_0^{-1} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ \alpha(\boldsymbol{z}) & 1 \end{pmatrix}, \quad (5)$$

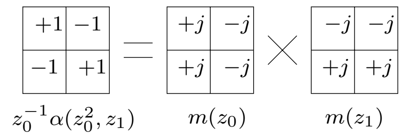


Fig. 8. The separable decomposition of $z_0^{-1}\alpha(z_0^2, z_1)$. The values of the 2-D Fourier transform of the filters are shown in the figure.

where $\alpha(\boldsymbol{z})$ is a free design parameter. It is easy to verify that the perfect reconstruction condition (3) is structurally guaranteed for arbitrary choice of $\alpha(\boldsymbol{z})$. Now substituting (4) into (1), we will get

$$H_0(\boldsymbol{z}) = \frac{1 + z_0^{-1}\alpha(z_0^2, z_1)}{\sqrt{2}} \quad (6)$$

Similarly, we can also write down $H_1(\boldsymbol{z}), G_0(\boldsymbol{z})$ and $G_1(\boldsymbol{z})$. Actually, they are all related to $H_0(\boldsymbol{z})$ as follows.

$$H_1(\boldsymbol{z}) = z_0\left(\sqrt{2} - \left(\sqrt{2}H_0(\boldsymbol{z}) - 1\right)H_0(\boldsymbol{z})\right) \quad (7)$$

$$G_0(\boldsymbol{z}) = -z_0^{-1}H_1(-z_0, z_1) \quad (8)$$

$$G_1(\boldsymbol{z}) = z_0^{-1}H_0(-z_0, z_1). \quad (9)$$

If $H_0(\boldsymbol{z})$ is the ideal filter with the desired checkerboard-shaped frequency support shown in Fig.3, i.e. if its Fourier transform takes the constant value $\sqrt{2}$ in the passband and 0 in the stopband, we can then verify from (7) - (9) that the other three filters $H_1(\boldsymbol{z}), G_0(\boldsymbol{z})$ and $G_1(\boldsymbol{z})$ will also achieve the desired frequency response. Therefore, we only need to design $H_0(\boldsymbol{z})$ to approximate the ideal filter on the checkerboard support. In turn, this implies that the Fourier transform of the filter $z_0^{-1}\alpha(z_0^2, z_1)$ must take constant values $(-1, 1, -1, 1)$ in the four quadrants of the 2-D frequency plane, as illustrated in Fig.8. Since this is a separable filter, we can decompose it as the product of two 1-D filters $m(z_0)$ and $m(z_1)$, i.e.

$$z_0^{-1}\alpha(z_0^2, z_1) = m(z_0) \cdot m(z_1). \quad (10)$$

If we further constrain $m(z)$ to have real coefficients, the only choice is

$$m(e^{j\omega}) = \begin{cases} -j, & \text{for} \quad \omega \in (0, \pi], \\ +j, & \text{for} \quad \omega \in (-\pi, 0]; \end{cases} \quad (11)$$

Meanwhile, the decomposition form in (10) also implies that

$$\alpha(\boldsymbol{z}) = z_1^{-1}\beta(-z_0)\beta(-z_1^2), \quad (12)$$

for some 1-D filter $\beta(z)$, and $m(z) = z^{-1}\beta(-z^2)$. From (11), we want $\beta(z)$ to be an allpass filter specified as:

$$|\beta(e^{j2\omega})| = 1, \forall\omega \quad (13)$$

$$\angle\beta(e^{j2\omega}) = 0.5\omega, \qquad \text{for} \quad \omega \in (-\pi, \pi); \quad (14)$$

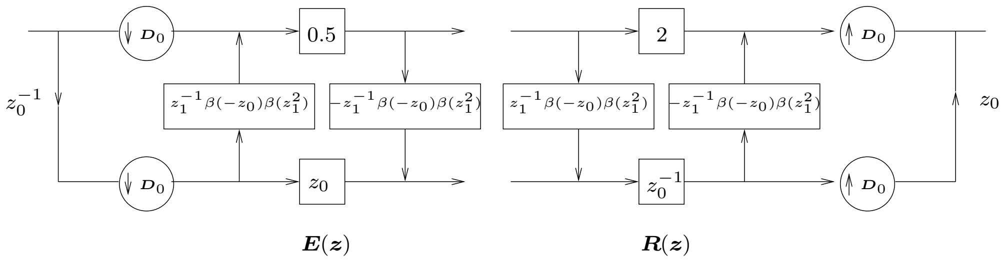


Fig. 9. The efficient polyphase implementation of the 2-D checkerboard filter bank.

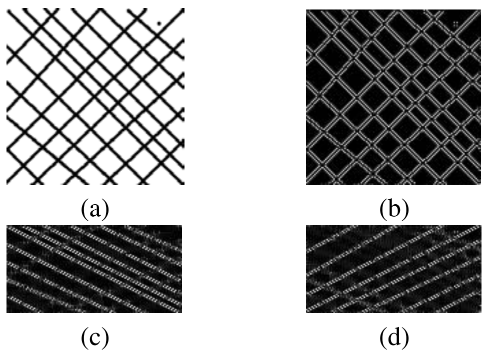


Fig. 10. (a) A synthetic image consisting of diagonal lines. (b) The diagonal subband of the wavelet transform. (c) Subband 5 of the proposed system. (d) Subband 6 of the proposed system.

In summary, the proposed filter bank design process is given as follows. We will first design some FIR filter $\beta(z)$ to approximate the conditions in (13) and (14). The analysis and synthesis filters H_0, H_1, G_0, G_1 can then be obtained from (12) and (6)-(9).

A nice property of the proposed filter design is that the structure is similar to a ladder network. In Fig. 9, we show the polyphase implementation of the filter bank. Although the designed filters are nonseparable, their polyphase components are separable and thus allow for a very efficient 1-D implementation.

4. NUMERICAL RESULTS

We apply the wavelet transform and the proposed transform on a synthetic image (Fig. 10(a)), which consists of lines oriented at both diagonal directions. Fig. 10(b) shows the diagonal subband (HH) of the wavelet transform. As discussed before, 45° and 135° directions are mixed in this subband and wavelets cannot discriminate between them. In Fig.10(c) and Fig.10(d), we show the subbands 5 and 6 of the proposed finer directional transform. Clearly the frequency mixing problem is solved, and the two subbands correctly capture the corresponding directional information.

5. CONCLUSION

In this work, we constructed a new transform that equips the wavelet transform with finer directionality. The filter bank construction of the transform is a concatenation of the separable wavelet transform with checkerboard filter banks. In 2-D, the new transform provides one lowpass subband and six directional highpass subbands at each scale. Just like the wavelet transform, the proposed transform is nonredundant, can be easily extended to higher dimensions, and has an efficient 1-D implementation. With the increased directionality, the proposed system can be an attractive tool for certain image processing applications, such as feature extraction, and classification. Detailed numerical results showing the performance of the system will be reported in a forthcoming paper.

6. REFERENCES

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