A FACTORIZATION PROCEDURE FOR FIR PARAUNITARY MATRICES

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ABSTRACT

A constructive factorization procedure for a polynomial paraunitary matrix as a product of atomic paraunitary matrices, each of lowest degree is given. The generic form of these paraunitary factors results from the use of projection matrices. In the univariate case each factor is of degree 1. In the multivariate case the generic factor is constructed based on the assumption of its existence .

Key words: Multivariate polynomial paraunitary matrix factorization, projection operators.

1. INTRODUCTION

For a discursive discussion on factorization of a FIR ($n \times n$) univariate paraunitary matrix with determinant z^{-N} as a product of N paraunitary matrices, each having determinant 1, see [1]. Factorizations using a dyadic-based basic building block and rotation-based (Householder, Givens) basic building blocks lead to structures with different finite arithmetic properties. While a basic factor is parameterized by a unit norm vector and has a generic form, the construction of the factorization from a specified polynomial paraunitary matrix involves some calculations. The procedure presented here not only reduces the computational chore but also makes the extraction of each factor routine. Significantly, this procedure generalizes to multivariate FIR paraunitary matrices and extracts the generic factor provided it exists. For the sake of brevity, the basic problem is enunciated in the bivariate case.

A square matrix $A(z_1, z_2) \in \mathbb{C}^{n \times n}[z_1, z_2]$ is paraunitary provided

$$A(z_1, z_2)A^*(\bar{z_1}^{-1}, \bar{z_2}^{-1}) = I_n$$

where I_n is the identity matrix of order n and the star superscript denotes complex conjugate transpose (hermitianconjugate) and the bar superscript denotes complex conjugate. Therefore,

$$\det A(z_1, z_2) = \pm z_1^{-l_1} z_2^{-l_2}$$

where l_1, l_2 are integers. The problems are:

P1 to decompose $A(z_1, z_2)$ as a product of lower degree "atomic" paraunitary matrices, where an atomic paraunitary matrix cannot be decomposed any further and

P2 parameterize, if possible, the atomic matrices in such a way that filter bank realization is facilitated (when $n = 2$, for example, it is well known that a univariate paraunitary matrix can be factored as a cascade of degree one lattices).

According to the procedure of Guiver and Bose [2], a nonsingular matrix $F(z_1, z_2) \in Q^{m \times m}[z_1, z_2]$, where Q is an arbitrary but fixed field, whose determinant has the irreducible factorization

$$\det F(z_1, z_2) = \prod_{i=1}^{k} f_i(z_1, z_2)$$

can constructively be factored as

$$F(z_1, z_2) = \prod_{i=1}^{k} F_i(z_1, z_2),$$
$$\det F_i(z_1, z_2) = f_i(z_1, z_2),$$

through, importantly, computations only in the base field Q,where $F_i(z_1, z_2) \in Q^{m \times m}[z_1, z_2]$ is an atomic factor of $F(z_1, z_2)$. This factorization is unique up to unimodular orthogonal matrices. With the imposition of the paraunitary constraint on $F(z_1, z_2)$, the application of the Guiver-Bose algorithm may not yield all paraunitary factors $F_i(z_1, z_2)$, for $i = 1, 2, \ldots, k$. The question then becomes if an atomic factorization involving only paraunitary factors is obtainable from the initial factorization. It is noted that the method in [2] does not generalize to the n-variate case, $n > 2$, because $\mathbb{C}^{n \times n}[z_1, z_2, ..., z_n]$ is not a determinantal factorization domain when $n > 2$ [3, ch 5].

It is well known that in the univariate case, the parametrization,

$$\prod_i (I - \mathbf{v_i}\mathbf{v_i^*}(1 - z^{-1}))$$

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for the unit norm vectors v_i is complete. The parameterized family of counterexample in [4] implies that the set of products of degree 1 non-commutative atomic factors in $\prod_i (I - \mathbf{v_i}\mathbf{v_i^*}(1 - z_1^{-1}))$ and $\prod_i (I - \mathbf{u_i}\mathbf{u_i^*}(1 - z_2^{-1}))$ (where v_i's and u_i's are unit norm vectors) is not complete.

2. PROJECTION OPERATOR ON THE IMAGE OF A SINGULAR MATRIX

Let M be a singular matrix of $M_n(K)$(or not of full rank in $M_{nm}(K)$) of rank $r < n$. The image of M forms a subspace W of K^n . There is a bounded linear map P satisfying $P^2 = P$ from K^n onto W and P is called a projection operator [5].The columns of P are the projections of the standard basis vectors and W is the image of P. Form $T = MM^*$, where M^* is the hermitian conjugate of M. Note that T is a self-adjoint operator and $Im(T) = Im(M)$. Now, define the characteristic polynomial associated with T, as $\chi_T(x)$. M, and hence T, being of rank r implies that $\chi_T(x) = x^{(n-r)}Q(x)$, where $Q(0) \neq 0$. Then $\frac{Q(T)}{Q(0)}$ is the projection operator on T's kernel and $P = I - \frac{Q(T)}{Q(0)}$ is the projection operator on T's image. A square matrix P is an orthogonal projection matrix iff $P^2 = P = P^*$, where P^* denotes the adjoint of the matrix P. In particular, for all matrices A and B such that $Im(A) \subseteq Im(M)$ and $Im(B) \subseteq Im(M)^\perp$

$$PA = A \quad , \quad PB = 0$$

3. APPLICATION TO 1D PARAUNITARY POLYNOMIAL MATRICES

Let $A(z^{-1})$ be a $(n \times n)$ paraunitary polynomial matrix of exact degree d (i.e. $A_d \neq 0$) in :

$$A(z^{-1}) = \sum_{i=0}^{d} z^{-i} A_i \quad , \quad A^*(\bar{z}^{-1}) = \sum_{i=0}^{d} z^{-i} A_i^*$$

The paraunitary property yields $A^*(z^{-1})A(z) = I$, from which we can derive a system of quadratic equations in terms of the coefficient matrices. One of these equations is $A_0^* A_d = 0$. Since $A_d \neq 0$, then $rank(A_0) < n$ and $A_0 \perp A_d$.

Proposition 1 *If P is the projection matrix associated with A_0, then $P + (I - P)z^{-1}$is a paraunitary left factor of the paraunitary matrix $A(z^{-1})$.*

Proof:

The paraunitariness of $P + (I - P)z^{-1}$ follows from :

$$\begin{aligned}
&(P + (I - P)z^{-1})^*(P + (I - P)z) \\
&= (P^* + (I - P^*)z^{-1})(P + (I - P)z) \\
&= P^2 + Pz - P^2 z + Pz^{-1} - P^2 z^{-1} + I - 2P + P^2 \\
&= P + Pz - Pz + Pz^{-1} - Pz^{-1} + I - 2P + P \\
&= I
\end{aligned}$$

The inverse of this paraunitary factor is evidently $P + (I - P)z$.

Furthermore $P + (I - P)z^{-1}$ is a left factor of $A(z^{-1})$ as justified next Let $A(z^{-1})$ is $A = A_0 + A_1 z^{-1} + ... + A_d z^{-d}$. Then we can factor $P + (I - P)z^{-1}$ and compute the remainder by left-multiplying $A(z^{-1})$ with the inverse factor $P + (I - P)z$.

$$\begin{aligned}
&(P + (I - P)z)A(z^{-1}) \\
&\qquad = PA_0 + PA_1 z^{-1} + ... + PA_d z^{-d} \\
&\qquad + A_0 z + A_1 + ... + A_d z^{-d+1} - \\
&\qquad - PA_0 z - PA_1 - ... - PA_d z^{-d+1}
\end{aligned}$$

Since $PA_0 = A_0$, we can compute the term with positive power of z, as

$$A_0 z - PA_0 z = 0$$

Moreover since $PA_d = 0$, therefore the remainder is a polynomial matrix in z^{-1} of reduced degree $d - 1$.

This leads to a very simple and powerful factorization procedure for all 1D paraunitary matrices, by iteratively repeating this factor extraction step, until the remaining matrix is a polynomial paraunitary matrix of degree one.

Comments :

- The computation of a factor only depends on the constant term of the matrix.
- Contrary to most factorization methods, this procedure does not act by reducing the degree of the matrix' determinant but the degree of the matrix' elements. Therefore our final factors won't necessarily be of determinant one as in Vaidyanathan's method or Givens' rotations or the Householder decomposition [1]. This can only occur if the polynomial degree of the elements is strictly less than the degree of the determinant which happen when factors commute with each other.

Example 1 *Application to the two-channel case.*

In the two-channel filterbank case, the computation of the projection matrix is simplified as:

$$P = \frac{A_0 A_0^*}{Tr(A_0 A_0^*)},$$

which has hermitian symmetry and is, equivalently, orthogonal. Indeed the charateristic polynomial becomes : $\chi_T(x) = x(x - Tr(T))$ *and then* $Q(T) = T = Tr(T)$, $Q(0) = Tr(T)$,

$$P = I - \frac{Q(T)}{Q(0)} = \frac{T}{Tr(T)}$$

Example 2 *A simple example for the four-channel case*

$$A = \begin{pmatrix} \frac{1+z^{-1}}{2} & \frac{1-z^{-2}}{4} & \frac{1-z^{-1}-z^{-2}+z^{-3}}{8} & \frac{1-3z^{-1}+3z^{-2}+-z^{-3}}{8} \\ \frac{1-z^{-1}}{2} & \frac{1+2z^{-1}+z^{-2}}{4} & \frac{1+z^{-1}-z^{-2}-z^{-3}}{8} & \frac{1-z^{-1}-z^{-2}+z^{-3}}{8} \\ 0 & \frac{1-z^{-1}}{2} & \frac{1+2z^{-1}+z^{-2}}{4} & \frac{1-z^{-2}}{4} \\ 0 & 0 & \frac{1-z^{-1}}{2} & \frac{1+z^{-1}}{2} \end{pmatrix}$$

Let $P^{(k)}, L^{(k)} and R^{(k)}$ *denote respectively, the projection matrix, the associated left-factor and remainder at the* k^{th} *iteration. Then,*

$$P^{(1)} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{(1)} = \begin{pmatrix} \frac{1+z^{-1}}{2} & \frac{1-z^{-1}}{2} & 0 & 0 \\ \frac{1-z^{-1}}{2} & \frac{1+z^{-1}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+z^{-1}}{2} & \frac{1-z^{-2}}{4} & \frac{1-2z^{-1}+z^{-2}}{4} \\ 0 & \frac{1-z^{-1}}{2} & \frac{1+2z^{-1}+z^{-2}}{4} & \frac{1-z^{-2}}{4} \\ 0 & 0 & \frac{1-z^{-1}}{2} & \frac{1+z^{-1}}{2} \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+z^{-1}}{2} & \frac{1-z^{-1}}{2} & 0 \\ 0 & \frac{1-z^{-1}}{2} & \frac{1+z^{-1}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+z^{-1}}{2} & \frac{1-z^{-1}}{2} \\ 0 & 0 & \frac{1-z^{-1}}{2} & \frac{1+z^{-1}}{2} \end{pmatrix}$$

$A = L^{(1)} L^{(2)} R^{(2)}$ *and the algorithm terminates because the remainder* $R^{(2)}$ *is an atomic paraunitary factor.*

4. EXTENSION TO THE MULTIVARIATE CASE

We present here how this factorization algorithm can be extended to the factorization of non-separable multivariate matrices when possible. Indeed, as shown by Park [4] the complete factorization into paraunitary matrices is not always possible, even if a general factorization exist [2]. If the paraunitary factorization exists, it can be obtained from Guiver and Bose's factorization by the introduction of well-chosen unitary matrices in between the factors.

But we can also again directly apply the projection operator factorization. The condition for which the factorization is possible emerges rather quickly from the algorithm and it is interesting to notice that it comes back to the same condition necessary to convert the Guiver and Bose's factorization into a paraunitary factorization.

For the bivariate case, suppose that $A(z_1^{-1}, z_2^{-1})$ is of the form:

$$A(z_1^{-1}, z_2^{-1}) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} A_{i,j} z_1^{-i} z_2^{-j}$$

Proposition 2 *A polynomial paraunitary matrix in* z_1^{-1} *(resp.* z_2^{-1}*) of degree* $(1,0)$ *(resp.* $(0,1)$*) and such that* $rank(A_{00}) = m-1$*, can be left-factored from A if and only if all the matrices* A_{0j} *ie* A_{01},...,A_{0d_2} *(resp.* A_{i0}*) are included in the image of* A_{00}.

Proof

- $\Rightarrow$ Suppose $A(z_1^{-1}, z_2^{-1}) = L(z_1^{-1}) A'(z_1^{-1}, z_2^{-1})$ where $L(z_1^{-1})$ is of degree 1 in z_1^{-1}. Define $L(z_1^{-1}) = L_0 + L_1 z^{-1}$. Then, in particular

  $$A_{0j} = L_0 A'_{0j} \qquad \forall j \in \{0, 1, ..., d_2\}$$

  so either A_{0j} is null or *Im*$(A_{0j}) \subseteq$ *Im*(L_0) for any $j \in \{0, 1, ..., d_2\}$. Therefore, all non-zero matrices A_{0j} have same image.

- $\Leftarrow$ Suppose now *Im*$(A_{0j}) \subseteq$ *Im*(A_{00}) for all j. Then define $L(z_1^{-1}) = P + (I - P) z_1^{-1}$ where P is the projection operator associated with A_{00},

  $$\begin{aligned} & L^{-1}(z_1^{-1}) A(z_1^{-1}, z_2^{-1}) \\ & \quad = PA_{00} + PA_{01} z_2^{-1} + ... + PA_{0d_2} z_2^{-d_2} \\ & \quad + ... + PA_{d_1 d_2} z_1^{-d_1} z_2^{-d_2} \\ & \quad + A_{00} z_1 + A_{01} z_1 z_2^{-1} + ... + A_{0d_2} z_1 z_2^{-d_2} \\ & \quad + ... + A_{d_1 d_2} z_1^{-d_1+1} z_2^{-d_2} - \\ & \quad - PA_{00} z_1 - PA_{01} z_1 z_2^{-1} - ... - PA_{0d_2} z_1 z_2^{-d_2} \\ & \quad - ... - PA_{d_1 d_2} z_1^{-d_1+1} z_2^{-d_2} \end{aligned}$$

There are no term with positive power in z_2. The term with positive power in z_1, after recombining, is

$$z_1[(A_{00} - PA_{00}) + (A_{01} - PA_{01})z_2^{-1} + ... + (A_{0d_2} - PA_{0d_2})z_2^{-d_2}] = 0$$

since $PA_{0j} = A_{0j} \quad \forall j$. Finally, we consider the terms in $z_1^{-d_1}$ which are of the form $z_1^{-d_1}[PA_{d_1 0} + ... + PA_{d_1 d_2} z_2^{-d_2}]$. Once again the paraunitariness of $A(z_1^{-1}, z_2^{-1})$ gives a set of equations among which :

$$\begin{cases} A_{00}^* A_{d_1 d_2} = 0 \\ A_{00}^* A_{d_1 d_2 - 1} + A_{01}^* A_{d_1 d_2} = 0 \\ \vdots \\ A_{00}^* A_{d_1 0} + A_{01}^* A_{d_1 1} + ... + A_{0d_2}^* A_{d_1 d_2} = 0 \end{cases}$$

From this it is easy to show that $A_{d_1 j} \perp A_{00} \quad \forall j$ and equivalently $PA_{d_1 j} = 0 \quad \forall j$. Thus the remainder is indeed a paraunitary polynomial matrix of reduced degree $d_1 - 1$ in z_1^{-1}.

This result can be readily extended to the n-variate case as long as $Im(A_{0,k_2,...,k_n}) \subseteq Im(A_{00...0}) \quad \forall(k_2, ..., k_n)$. Then we can proceed to extract the factor $P + (I - P)z_1^{-1}$ where P is the projection matrix associated with the constant term matrix $A_{00...0}$.

Remark 1 *The bivariate case of degree $(1, d)$ or $(d, 1)$ has been studied by both Park [4] and Vaidyanathan [6]. Here is a simpler proof of the result stating that the atomic factorization is always possible for those types of matrices. Consider once again a subset of the paraunitariness equations, valid for the $(1, d)$ case:*

$$\begin{cases} A_{00}^* A_{1d} = 0 \\ A_{10}^* A_{0d} = 0 \\ A_{00}^* A_{0d} + A_{10}^* A_{1d} = 0 \end{cases}$$

Clearly if no matrix is null, then $A_{00} \perp A_{1d}$ and $A_{10} \perp A_{0d}$, consequently

- *either $A_{00}^* A_{0d} = 0$, i.e. $A_{00} \perp A_{0d}$ implying that $\mathrm{Im}(A_{00}) = \mathrm{Im}(A_{10})$, so that a left-factor in z_2^{-1} can be extracted*
- *or $A_{00}^* A_{0d} \neq 0$, and since A_{00} and A_{10} have to be of $rank 1$, therefore $\mathrm{Im}(A_{00}^*) = \mathrm{Im}(A_{10}^*)$, so that a right-factor in z_2^{-1} can be extracted*

The procedure for extraction of a factor (left or right) in z_2^{-1} can be recursively implemented until the remainder becomes the atomic factor in z_1^{-1}. Any special case with null matrices is trivial and leads to the same result. The argument is similar for the $(d, 1)$ case.

5. CONCLUSION

A procedure based on the use of projection operator is advanced for the factorization of a prescribed square polynomial paraunitary matrix as a product of atomic paraunitary factors. In the univariate case, this procedure is simpler than existing procedures, where the corresponding projection matrix is restricted to be of rank 1 and is difficult to construct. It is conjectured that there exists at least one bivariate polynomial paraunitary matrix of arbitrary but fixed degree that is an atomic (irreducible) factor. Therefore, there cannot be a canonic realization structure like the cascade of lattices (in the 1-D two channel case) for the 2-D two channel case. Due to the use of nonsquare bivariate polynomial matrices in signal processing and control, the role of paraunitariness in such a setting needs to be investigated.

6. REFERENCES

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