IMPROVEMENT OF MULTI-MODE-PCA BASED FILTERING USING FOURTH ORDER CUMULANTS

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ABSTRACT

The main idea of the paper is to replace the covariance matrix involved in TUCKALS3 algorithm with the fourth order cumulant slice matrix of the data tensor n-mode vectors in order to eliminate the Gaussian components of the additive noise. The good qualitative results of this new multi-mode filtering method are shown in the case of correlated noise reduction in a color image and a polarized seismic wave, modelled by third order tensors.

1. INTRODUCTION

In the context of an additive multidimensional *white* Gaussian noise, a new concept of multi-mode (MM) filtering of multidimensional and multicomponent signals, such as color or multispectral images, or polarized seismic waves, modelled by higher order tensors [1], has been proposed in [6, 8, 9].

In these studies, the measurement of a multidimensional signal $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \cdots \times I_N}$ by multicomponent sensors with additive *white* Gaussian noise $\mathcal{N}^w$, statistically independent from the signal, results in data tensor: $\mathcal{R} = \mathcal{X} + \mathcal{N}^w$. The estimation of desired signal $\mathcal{X}$ thanks to a MM filtering of noisy data tensor $\mathcal{R}$ [8, 9] can be written as follows:

$$\widehat{\mathcal{X}} = \mathcal{R} \times_1 H^{(1)} \cdots \times_N H^{(N)}, \tag{1}$$

in which, each n-mode of a N^{th} order noisy data tensor $\mathcal{R}$ is filtered by a matrix $H_n \in \mathbb{R}^{I_n \times I_n}$ called n-mode filter by means of a multilinear algebra operator called "n-mode product", and denoted by $\times_n$ [3].

In these methods, in extension to the classical matrix filtering methods [4] to data tensors, n-mode filters H_n are imposed to be projectors on the K_n-dimensional n-mode signal subspace. Their determination lies either on the rank-$(K_1, \ldots K_N)$ troncature of the Higher Order SVD (HOSVD-$(K_1, \ldots K_N)$) or on lower rank-$(K_1, \ldots K_N)$ tensor approximation (LRTA-$(K_1, \ldots K_N)$) [2, 3], both of which are based on TUKER3 tensor decomposition [10] that generalises the matrix SVD. The achievement of LRTA-$(K_1, \ldots K_N)$ needs a numerical iterative process based on an Alternative Least Square (ALS) optimization, called TUCKALS3 algorithm [5]. By construction, the LRTA-$(K_1, \ldots K_N)$ is a process that enables to achieve a multi-mode (MM) Principal Component Analysis (PCA) [5], and the HOSVD-$(K_1, \ldots K_N)$ can be considered as an approximation of the LRTA-$(K_1, \ldots K_N)$ since it consists of the initialization step of TUCKALS3 algorithm [3, 5].

In this paper, we propose to improve the MM-PCA based filtering in the case of an additive multidimensional *correlated* Gaussian noise, thanks to the fourth order cumulants, in order to eliminate the white Gaussian component of the noise. In section 2, we present TUCKALS3 algorithm on which lies the MM-PCA based filtering, and stress how this algorithm is based on second order statistics. In section 3 an improvement of the multi-mode filtering is proposed by replacing covariance matrix involved in TUCKALS3 algorithm by the associated fourth order cumulant. The good performances of this improved MM-PCA based filtering with fourth order cumulants are shown in section 4, in the case of additive *correlated* Gaussian noise.

2. RECALLS ON MM PCA

2.1. TUCKALS3 algorithm for LRTA-$(K_1, \ldots, K_N)$ and MM PCA

The TUCKALS3 Alternative Least Square algorithm [5] that can be summarized in the following steps:

1. initialisation $k = 0$:
   Initial projectors $P_0^{(n)}$ are obtained thanks to tensor $\mathcal{R}$ HOSVD truncation [3, 5]. $\forall n = 1, \ldots N$, $P_0^{(n)} = U_0^{(n)} U_0^{(n)^T}$. $U_0^{(n)}$ is the matrix of the K_n left singular vectors associated with the K_n largest singular values of tensor $\mathcal{R}$ n-mode unfolding matrix R_n [3].
2. ALS loop: while $\|\mathcal{R} - \mathcal{B}_k\|_F^2 > \epsilon$, $\epsilon > 0$ prior fixed threshold,
   (a) for $n = 1$ to N:

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i. $\mathcal{B}^{(n),k} = \mathcal{R} \times_1 P_{k+1}^{(1)} \cdots \times_{n-1} P_{k+1}^{(n-1)} \times_{n+1} P_k^{(n+1)} \times_N P_k^{(N)}$;

ii. n-mode unfold $\mathcal{B}^{(n),k}$ into matrix $B_n^{(n),k} = R_n(P_{k+1}^{(1)} \otimes \cdots P_{k+1}^{(n-1)} \otimes P_k^{(n+1)} \cdots \otimes P_k^{(N)})$;

iii. compute matrix $C^{(n),k} = B_n^{(n),k} R_n^T$;

iv. process $C^{(n),k}$ EVD, put the K_n eigenvectors associated with the K_n largest eigenvalues into $U_{k+1}^{(n)} \in \mathbb{R}^{I_n \times K_n}$;

v. compute $P_{k+1}^{(n)} = U_{k+1}^{(n)} U_{k+1}^{(n)^T}$;

(b) compute $\mathcal{B}_{k+1} = \mathcal{R} \times_1 P_{k+1}^{(1)} \cdots \times_N P_{k+1}^{(N)}$, $k \leftarrow k+1$.

3. output: $\mathcal{B}_{k_{stop}} = \mathcal{R} \times_1 P_{k_{stop}}^{(1)} \cdots \times_N P_{k_{stop}}^{(N)}$, the best lower rang-$(K_1, \ldots, K_N)$ approximation of $\mathcal{R}$.

The symbol $\otimes$ denotes the Kronecker product.

2.2. Second order statistics in TUCKALS3 algorithm

It is possible to give a statistical sense to matrix $C^{(n),k}$ from step 2(a)iii. Let's define by $\mathbf{b}_j^{(n),k}$, $j = 1, \ldots M_n$, with: $M_n = I_1 \cdots I_{n-1} I_{n+1} \cdots I_N$, the n-mode vectors of tensor $\mathcal{B}^{(n),k}$, i.e. n-mode unfolding matrix $B_n^{(n),k}$ column vectors. Let's define as well by $\mathbf{r}_j^{(n)}$, $j = 1, \ldots M_n$, the n-mode vectors of tensor $\mathcal{R}$. Matrix $C^{(n),k} = B_n^{(n),k} R_n^T$ can be written as: $C^{(n),k} = [\mathbf{b}_1^{(n),k}, \ldots, \mathbf{b}_{M_n}^{(n),k}][\mathbf{r}_1^{(n)}, \ldots, \mathbf{r}_{M_n}^{(n)}]^T = \sum_{j=1}^{M_n} \mathbf{b}_j^{(n),k} \mathbf{r}_j^{(n)^T}$.

As a consequence, up to the multiplicative factor $\frac{1}{M_n}$, matrix $C^{(n),k}$ is an estimation of the covariance matrix between data tensor $\mathcal{R}$ n-mode vectors and tensor $\mathcal{B}^{(n),k}$ n-mode vectors.

Considering, in the previous expression of $C^{(n),k}$, that $\{\mathbf{r}_j^{(n)}, j = 1, \ldots M_n\}$, and $\{\mathbf{b}_j^{(n),k}, j = 1, \ldots M_n\}$ are the M_n realizations of two random vectors $\mathbf{r}^{(n)}$ and $\mathbf{b}^{(n),k}$ associated respectively with the n-mode vectors of data tensors $\mathcal{R}$ and $\mathcal{B}^{(n),k}$, matrix $C^{(n),k}$ can be written as a second order moment: $C^{(n),k} = \mathrm{E}[\mathbf{b}^{(n),k} \mathbf{r}^{(n)^T}]$, where $\mathrm{E}[\cdot]$ denotes the expectation operator.

Moreover, the K_n eigenvectors associated with the largest eigenvalues of matrix $C^{(n),k}$ from which projector $P_{k+1}^{(n)}$ is built (at step 2(a)v) are the mutual principal components of tensor $\mathcal{R}$ and tensor $\mathcal{B}^{(n),k}$ n-mode vectors. Thus the LRTA-$(K_1, \ldots, K_N)$ of data tensor $\mathcal{R}$ consists of a principal component analysis processed simultaneously on every n-mode.

3. IMPROVEMENT OF THE MM FILTERING USING FOURTH ORDER CUMULANT

In practice the noise whiteness and gaussianity is not always fulfilled. The use of higher order statistics consists of a classical means to eliminate the noise Gaussian components [7].

3.1. Fourth order cumulants

As remarked in section 2.2, matrix $C^{(n),k}$, at step 2(a)iii and TUCKALS3 algorithm k^{th} iteration, is defined as a second order moment. It can be replaced by a fourth order cumulant [7]: $\mathbf{C}^{(n),k} = \mathrm{Cum}(\mathbf{b}^{(n),k}, \mathbf{b}^{(n),k^T}, \mathbf{r}^{(n)}, \mathbf{r}^{(n)^T})$. In practice, in order to reduce the computational load, a cumulant slice matrix of $\mathbf{C}^{(n),k}$ can be computed. The cumulant slice matrix associated with the first component of vector $\mathbf{b}^{(n),k}$, is given by the following $(I_n \times I_n)$-hermitian matrix [7, 11]: $\mathbf{C}_\mathbf{1}^{(n),k} = \mathrm{Cum}(b_1^{(n),k}, b_1^{(n),k}, \mathbf{r}^{(n)}, \mathbf{r}^{(n)^T})$.

The generic (i,j)-term of cumulant slice $\mathbf{C}_\mathbf{1}^{(n),k}$ expressed with the expectation operator is: $\mathbf{C}_{\mathbf{1}ij}^{(n),k} = \mathrm{E}[b_1^{(n),k^2} r_i^{(n)} r_j^{(n)}] - 2\mathrm{E}[b_1^{(n),k} r_i^{(n)}]\mathrm{E}[b_1^{(n),k} r_j^{(n)}]$.

The practical estimation of $\mathbf{C}_\mathbf{1}^{(n),k}$ uses the M_n realizations of random vectors $\mathbf{r}^{(n)}$ and $\mathbf{b}^{(n),k}$. Defining by $b_{ij}^{(n),k}$ and $r_{ij}^{(n)}$ the (i,j)-term of $B_n^{(n),k}$ and R_n n-mode unfolding matrices, the estimation of $\mathbf{C}_{\mathbf{1}ij}^{(n),k}$ term is given by:

$$\begin{aligned} \mathbf{C}_{\mathbf{1}ij}^{(n),k} &= \tfrac{1}{M_n}(\textstyle\sum_{p=1}^{M_n} b_{1p}^{(n),k^2} r_{ip}^{(n)} r_{jp}^{(n)}) \\ &- \tfrac{2}{M_n^2}(\textstyle\sum_{p=1}^{M_n} b_{1p}^{(n),k} r_{ip}^{(n)})(\sum_{p=1}^{M_n} b_{1p}^{(n),k} r_{jp}^{(n)}) \end{aligned} \qquad (2)$$

Although cumulant slice matrix $\mathbf{C}_\mathbf{1}^{(n),k}$ may not contain strictly the same information as the whole cumulant tensor $\mathbf{C}^{(n),k}$, as shown in the next section simulations on figures 1(e) and 2(m), from an experimental point of view, the results given by the MM-PCA based filtering improved either by $\mathbf{C}_\mathbf{1}^{(n),k}$ or by $\mathbf{C}^{(n),k}$ do not present a large difference.

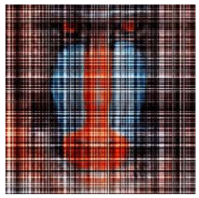
3.2. Proposed algorithm

When the cumulant slice matrix $\mathbf{C}_\mathbf{1}^{(n),k}$ is used, in step 2(a)iv of TUCKALS3 algorithm, the lower rank-K_n approximation of matrix $C^{(n),k}$, that leads to matrix $U_{k+1}^{(n)}$, is replaced by the one of $\mathbf{C}_\mathbf{1}^{(n),k}$.

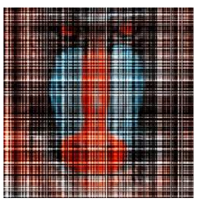
When the whole cumulant tensor $\mathbf{C}^{(n),k}$ is used, matrix $U_{k+1}^{(n)}$ is determined by computing tensor $\mathbf{C}^{(n),k}$ LRTA-(K_n, K_n, K_n, K_n), which generalize the matrix lower rank approximation [3], given by : $\mathbf{C}^{(n),k}(K_n, K_n, K_n, K_n) = \mathcal{S} \times_1 U_{k+1}^{(n)} \cdots \times_4 U_{k+1}^{(n)}$, where $\mathcal{S} \in \mathbb{R}^{K_n \times K_n \times K_n \times K_n}$ is the core tensor.

TUCKALS3 algorithm initialization step can also be modified by using the fourth order cumulant slice matrix. The initialization with the whole fourth order cumulant tensor can be obtained straightforwardly. For $n = 1, \ldots, N$, matrice $U_0^{(n)}$, with which initial projector $P_0^{(n)}$ is built, is the matrix of the K_n eigenvectors associated with hermitian matrix $R_n R_n^T$. Hence, it also consists of an estimation of data tensor $\mathcal{R}$ n-mode vector covariance matrix. In the proposed method, matrix $R_n R_n^T$ is replaced with $\mathbf{C}_{\mathbf{1}}^{(n),0}$, the fourth-order cumulant slice of data tensor $\mathcal{R}$ n-mode vectors, which is estimated thanks to relation (2) by replacing element $b_{1p}^{(n),k}$ by element $r_{1p}^{(n)}$, $\forall p = 1, \ldots, M_n$.

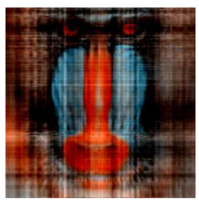
(a) initial image



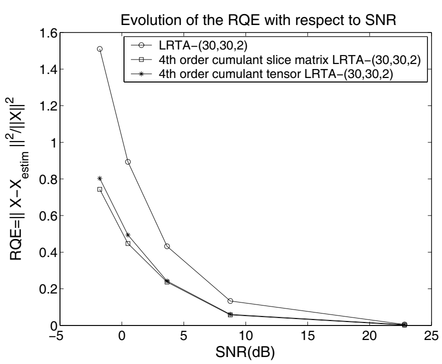
(b) SNR=-2.1dB



(c) LRTA-(30,30,2)



(d) LRTA$\mathbf{C_1}$-(30,30,2)



(e)

Fig. 1. 'Baboon' standard $(256 \times 256 \times 3)$-image

4. SIMULATION RESULTS

In the following simulations, the proposed method is applied for noise reduction in a color image and a polarized seismic wave, both of which can be modelled by a third order tensor $\mathcal{R} = \mathcal{X} + \mathcal{N}$. $\mathcal{N}$ is considered as a *correlated* Gaussian noise that can be written as $\mathcal{N} = \mathcal{N}^w \times_1 W^{(1)} \times_2 W^{(2)} \times_3 W^{(3)}$, where $\mathcal{N}^w$ is a *white* Gaussian noise $\mathcal{N}^w$, statistically independent from the signal, and $\forall n = 1, 2, 3$, $W^{(n)}$ are weighting matrices which make the correlation between the n-modes.

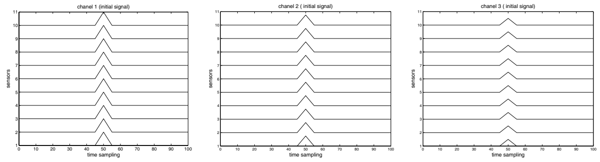
The estimated signal tensor is $\widehat{\mathcal{X}} = \mathcal{R} \times_1 P^{(1)} \times_2 P^{(2)} \times_3 P^{(3)}$, in which, $\forall n = 1, 2, 3$, $P^{(n)}$ is the orthogonal projector on the K_n dimensional n-mode signal subspace, obtained after convergence of the proposed improved TUCKALS3 algorithm thanks to the fourth-order cumulant slice matrix, noted by LRTA$\mathbf{C_1}$-(K_1, K_2, K_3). If the whole cumulant tensor is used, it is noted by LRTA$\mathbf{C}$-(K_1, K_2, K_3).

In order to a posteriori verify the estimated image quality we propose to use the Relative Quadratic Error criterion (RQE) defined thanks to the tensor Frobenius Norm [3] by: $RQE(\widehat{\mathcal{X}}) = \frac{\|\widehat{\mathcal{X}} - \mathcal{X}\|_F^2}{\|\mathcal{X}\|_F^2}$. The RQE enables a qualitative comparison between the classical LRTA-(K_1, K_2, K_3) and the proposed LRTA$\mathbf{C_1}$-(K_1, K_2, K_3) based MM filterings.

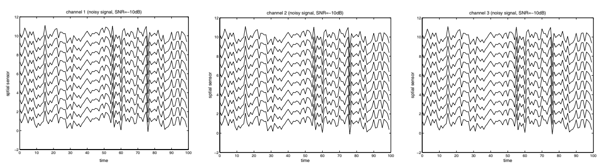
Color image - Let's consider "Baboon" standard color image (Fig. 1(a)) modelled by tensor $\mathcal{X} \in \mathbb{R}^{256 \times 256 \times 3}$. We suppose the initial n-mode ranks of this image known and fixed at (K_1, K_2, K_3)=$(30, 30, 2)$. Noisy image $\mathcal{R}$, in which *correlated* noise $\mathcal{N}$ is such that $SNR = 10log(\frac{\|\mathcal{X}\|_F^2}{\|\mathcal{N}\|_F^2}) = 2.1dB$, is represented in Fig. 1(b). The classical LRTA-$(30, 30, 2)$ based MM filtering of noisy image $\mathcal{R}$ results in estimated image represented in Fig. 1(c), in which the *correlated* noise has almost not been removed. The proposed LRTA$\mathbf{C_1}$-$(30, 30, 2)$ based MM filtering of $\mathcal{R}$ results in estimated image represented in Fig. 1(d), which has a greatly better quality compared to previous image 1(c). Finally, the evolution of the RQE with respect to the SNR(dB) for the LRTA$\mathbf{C_1}$, the LRTA$\mathbf{C}$ and the classical LRTA based MM filterings, is represented in Fig. 1(e). It shows that the NQE obtained with the proposed method is always lower than the one obtained with the classical MM filtering.

Polarized seismic Wave - We consider now, a polarized seismic plane wave whose wave-front is parallel to the antenna plane, modelled by tensor $\mathcal{X} \in \mathbb{R}^{10 \times 200 \times 3}$. The linear antenna is composed of 10 sensors, and the time sampling on each sensor represents 200 samples. The wave direction of propagation is supposed orthogonal to the antenna plane. The X, Y and Z-polarisation components are shown respectively on Figures 2(a)-2(c), and represent a triangular impulse with the same temporal length but different amplitude. The three polarisation components of noisy signal $\mathcal{R}$, in which *correlated* noise $\mathcal{N}$ is such that $SNR = -10dB$, are shown respectively on Figures 2(d)-2(f). As shown in Figures 2(g)-2(g), the classical LRTA-$(1, 1, 1)$ based MM filtering of $\mathcal{R}$ is not able to remove the *correlated* noise. On the opposite, as shown in Figures 2(j)-2(l), the proposed LRTA$\mathbf{C_1}$-$(1, 1, 1)$ based MM filtering of $\mathcal{R}$ leads to a perfect estimation of the initial signal. The good performances of the proposed LRTA$\mathbf{C_1}$-$(1, 1, 1)$ based MM filtering is confirmed by the evolution of the RQE with respect to the SNR(dB) shown in Fig 2(m). All along the SNR range, the RQE is null with the proposed method where as the RQE increases when the SNR decreases, for the classical method.

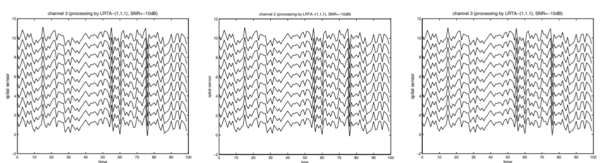
Note also that, for both examples, the RQEs obtained with the LRTA$\mathbf{C_1}$-(K_1, K_2, K_3) and with the LRTA$\mathbf{C}$-(K_1, K_2, K_3) based MM filtering do not present a large difference, whatever the SNR.



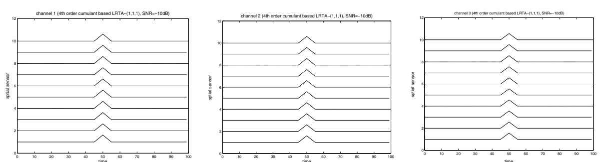
(a) channel 1 initial signal (b) channel 2 initial signal (c) channel 3 initial signal



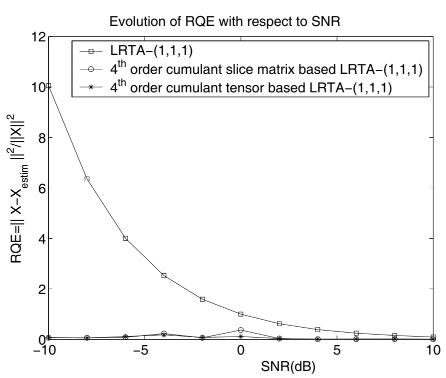
(d) correlated Gaussian noise (e) SNR=-10dB (f)



(g) LRTA-$(1, 1, 1)$ (h) LRTA-$(1, 1, 1)$ (i) LRTA-$(1, 1, 1)$



(j) LRTA$\mathbf{C_1}$-$(1, 1, 1)$ (k) LRTA$\mathbf{C_1}$-$(1, 1, 1)$ (l) LRTA$\mathbf{C_1}$-$(1, 1, 1)$



(m) RQE evolution with respect to SNR (dB)

Fig. 2. Polarized plane seismic wave for whose wave-front is parallel to the antenna plane.

5. CONCLUSION

The main idea of the paper is to replace the covariance matrix involved in TUCKALS3 algorithm with the fourth order cumulant slice matrix of the data tensor n-mode vectors in order to eliminate the Gaussian components of the additive noise and improve the classical MM filtering based on LRTA-$(K_1, \ldots, K_N)$. The good qualitative results of this new multi-mode filtering method are shown in the case of noise reduction in a color image and a polarized seismic wave.

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