PILE-UP CORRECTION ALGORITHMS FOR NUCLEAR SPECTROMETRY

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ABSTRACT

This article presents a problem encountered in nuclear physics, queuing theory and point processes. The studied signal consists of pulses of random length and energy, possibly sampled, whose time occurrences are points of an homogenous Poisson process. Incoming pulses can combine into pile-ups, which results into a biased estimation of the density of lengths and energies. We introduce a model based on two marked point processes and derive an analytical relation between the probability density function (pdf) of the observed pile-ups and the pdf of the pulses, that leads to an algorithm for pile-up correction, in both cases of a continuous time and discrete time signals. Simulations show cancellation of the pile-up effect and prove efficiency of the algorithms in gamma spectrometry.

1. INTRODUCTION

1.1. Description of the Type II Counter Problem

We consider a signal composed of pulses, each pulse being characterized by its length U, its energy E and its occurrence time T. We assume that occurrence times are the points of an homogenous Poisson process, and we are interested in the pdf of (U, E). However, since each pulse has a strictly positive length, some pulses can combine into pile-ups ; therefore, random variables U and E are not directly observable, instead we only have access to lengths L and energies M of pile-ups. This context is refered in [1] as the Type II Counter Problem ; Other references to this problem can be found in [2].

Figure 1 illustrates this problem : assume that T_n is the arrival time of the n-th pulse, U_n its length and E_n its energy. When the n-th pulse arrives, its energy is recorded, and thus we observe (U_n, E_n). On the other hand, the $n+2$-th pulse begins before the $n+1$-th pulse ends ; we cannot observe in that case neither (U_{n+1}, E_{n+1}) nor (U_{n+2}, E_{n+2}), but $(L_{n+1}, M_{n+1}) = (T_{n+2}+U_{n+2}-T_{n+1}, E_{n+1}+E_{n+2})$. We follow the terminology used in physics and call this phenomenon the *pile-up* effect. Our goal is to estimate the pdf

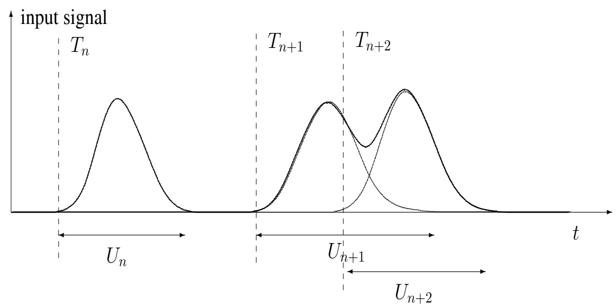

Fig. 1. Illustration of Type II Counter Problem : input signal with arrival times T_i and service times U_i, $i = n, \ldots, n+2$.

of (U, E), given N pile-ups $\{(l_i, m_i)\}_{1 \leq i \leq N}$ (the exact relation between pulses and pile-ups is given in [3]).

Type II Counter Problem is encountered in nuclear physics, particularly in γ or X spectrometry. In this application a radioactive source emits randomly photons following a Poisson process ; a dispositive based on charge collection changes the energy of a detected photon into an electrical pulse, whose integral is proportional to the energy of the photon. The physicist is interested in the distribution of the energy of the incoming photons, that is the pdf of E.

1.2. State of the art

The problem of pile-ups in the case of an homogenous Poisson process marked by a single random variable has been well studied in queuing theory where U_n represents the n-th busy period of a server ; results of [1] and [4] give a relation for the Laplace transform of the pdf of U_n and establish an estimate of the cumulative distribution function (cdf) of U_n, using the formalism of the queuing theory and the M/G/∞ model. However, the lengths of pulses are less used in spectrometry than their energies.

In papers dealing specifically with Type II Counter Problem, such as [5], we only consider the energy E_n as the n-th mark of the point T_n, and have results concerning the pdf of pile-ups $\{M_n\}$, but not the distribution of interest.

Our approach is motivated by the experimental observation that lengths and energies of the detected pulses are not independent, and therefore to combine the aspects of queuing theory with the spectrometry point of view. We can show that the pdf of the lengths and energies of pile-ups, denoted by ν, can be related to the pdf of the lengths and energies of pulses, denoted by μ.

1.3. Purpose and organization

This paper presents relations and algorithms to compensate the pile-up effect for both continuous-time and discrete-time signals. Section 2 introduces the model and an analytical formula between the two pdfs ; a formula for discrete signals is also presented. Section 2 gives the algorithms deduced from the main theorems. Simulations and results are presented in section 3.

2. ASSUMPTIONS AND MAIN RESULTS

2.1. Notations

Denote by λ the intensity of the underlying Poisson process. Denote by U_n, E_n, T_n respectively the length, the energy and the occurence time of the n-th pulse. We define the n-th idle period as the period between the end of the n-th pile-up and the beginning of the $n+1$-th pile-up. Denote by L_n, M_n, O_n respectively the length, the energy and the occurrence time of the n-th pile-up. Denote by $\bar{E}_t$ the cumulative energy at time t, that is :

$$\bar{E}_t = \sum_{\{i; T_i \in [0;t]\}} E_i.$$

We also assume there is no pulse at time $t = 0$.

2.2. Continuous case

Denote by s_t the value of the input signal at time t, and assume that $s_t > 0$ when at least a pulse occurs, and $s_t = 0$ in an idle period. The following theorem, which relates μ to ν, is proved by computing the probability $\mathbb{P}(s_t = 0\ ;\ \bar{E}_t \leq e)$ conditioning by the number of pulses in $[0;t]$, then conditioning by the number of pile-ups in $[0;t]$:

Theorem 2.1 *Under assumptions of Sections 2.1 and 2.2, we have :*

$$\int_0^\infty e^{-(\lambda+s)t} \exp\left(\lambda \int_0^\infty e^{-pv} k(t,v)\, dv\right)\, dt$$
$$= \left(s + \lambda - \lambda \int_{\mathbb{R}_+^2} e^{-st-pv} \nu(t,v)\, dt\, dv\right)^{-1}, \quad (1)$$

where

$$k(t,v) = \int_0^t \left\{\int_0^a \mu(u,v)\, du\right\}\, da.$$

Due to Theorem 2.1, we propose in Table 1 an algorithm based on equation (1) for pile-up correction in the continuous case.

(i) Given the observed samples, compute a kernel estimate of ν.
(ii) Compute the right term of (1).
(iii) Invert numerically the Laplace transforms in the left term of (1), given numerical samples from (ii) and deduce $k(\cdot,\cdot)$.
(iv) Deduce the pdf of interest from $k(\cdot,\cdot)$.

Table 1. Continuous version of the algorithm

Steps (i) to (iii) of the algorithm return the double cdf following the first dimension of μ. It seems natural to deduce the distribution of interest by a double numerical derivation in step (iv). However, since the integral is a compact operator, derivation (and *a fortiori* double derivation) is an ill-posed problem, that is its solution is very sensitive to initial conditions or observations. Another point is that an unconstrained step of double derivation could lead to an estimate which is not a probability density function (for example taking negative values). It is therefore necessary to solve both problems.

Denote by D the differentiation operator. In order to solve both problems (positivity and integration to 1), we replace the operation $D^2 k$ by the constrained optimization step :

$$\arg\min_{\mathbf{f}\geq\mathbf{0}} ||(D^{\#})^2 f - k||^2 \quad (2)$$

where $D^{\#}$ is the pseudo-inverse of D and $||.||$ is the euclidian norm. In practical applications, k is a matrix, so optimization (2) can be solved by using Lawson-Hanson algorithm. A normalization can be done after this step, to ensure the integration to 1 of the result.

2.3. Discrete case

In the case of sampled and quantified signals, we are not interested directly to μ but to its discrete version. The event $\{s_t = 0\ ;\ \bar{E}_t \leq e\}$ cannot be directly observed. It is therefore necessary to adapt notations. For convenience, the sampling period T_e will be set to 1.

Denote by s_n the value of the input signal at the n-th sampling time. As in 2.2 assume that $s_n > 0$ when at least a pulse occurs, and $s_n = 0$ in an idle period. We also assume there is no pulse at time $t = 0$. Denote by $\alpha = e^{-\lambda}$. We say that a transition occurs at time n if and only if $s_{n-1} = 0$ and $s_n > 0$. Remark that random variables introduced in 2.1 are now discrete.

Denote by b the probability distribution of the the pile-ups, and by p the probability distribution of the pulses, that is for all integers n and e :

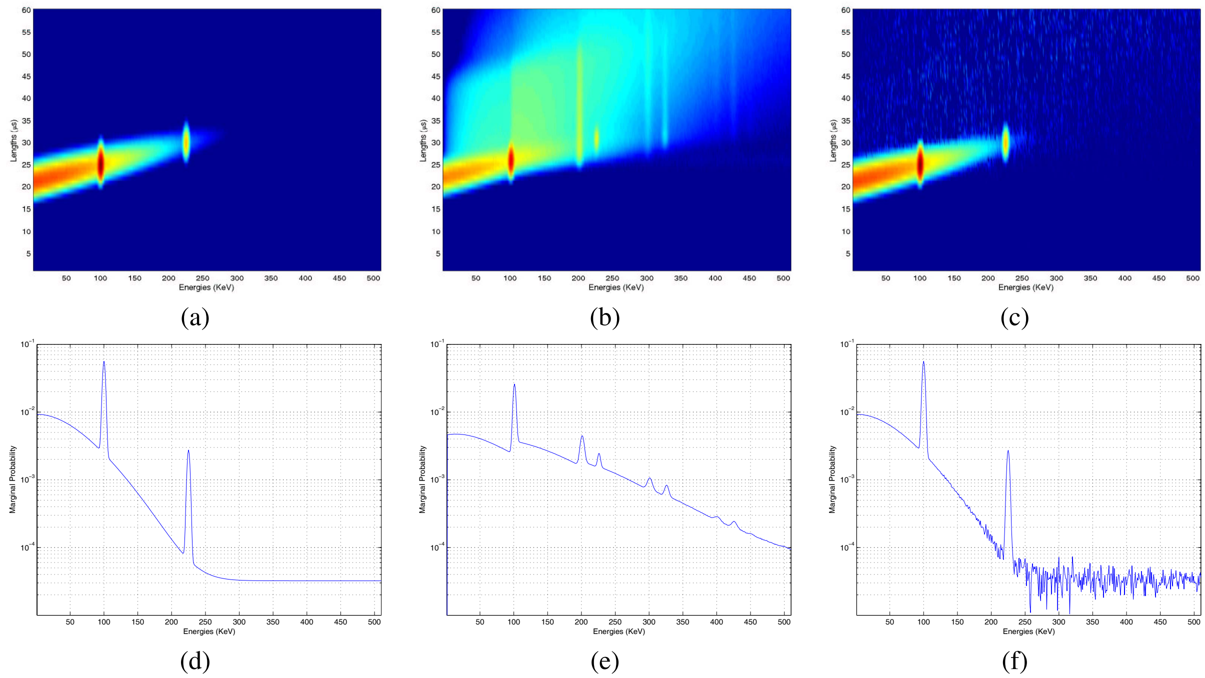

Fig. 2. Pile-ups and pile-up correction results in the discrete case

$$
\begin{aligned}
b_{n,e} &= \mathbb{P}(L = n \cap M = e) &(3)\\
p_{n,e} &= \mathbb{P}(U = n \cap E = e) &(4)
\end{aligned}
$$

Finally, denote by B the generating function associated with b. It is possible to express $\mathbb{P}(s_n = 0; \bar{E}_n = e)$ with respect to (3) using a renewal argument (see [6]). Moreover, this probability can be computed using (4) and usual properties of the Poisson process. These considerations lead to a theorem, which relates the generating functions of b and p :

Theorem 2.2 *Under assumptions of Section 2.3, we have :*

$$\sum_{n=0}^{\infty} z^n (\alpha^{n-K_n(s)}) = \frac{1}{1-(\alpha z + (1-\alpha) z B(z,s)))} \quad (5)$$

where

$$K_n(s) = \sum_{i=1}^{\infty} k_{n,i} s^i \;\; \textit{and} \;\; k_{n,e} \triangleq \sum_{j=1}^{n} \sum_{m=1}^{j-1} p_{m,e}.$$

We therefore propose in Table 2 an algorithm of pile-up correction in the discrete case based on equation (5). As in the continuous case, step (iv') of the algorithm is done by a constrained optimization algorithm and a normalization.

Proofs of Theorem 2.1 and Theorem 2.2 can be found in [3].

(i')	Given the observed samples, compute an histogram of b.
(ii')	Recursively compute the right term of (5) in power series.
(iii')	Compare the obtained power series with the left term of (5) and deduce $k_{n,e}$.
(iv')	Deduce the probability distribution from $k_{n,e}$.

Table 2. Discrete version of the algorithm

3. RESULTS AND DISCUSSION

3.1. Experimental protocol

We consider $N = 1.5 \times 10^8$ samples $\{(l_i, m_i)\}_{1 \le i \le N}$ obtained recursively from drawings of (U, E). In order to stay close to nuclear physics applications, the density μ is

$$\mu(x,y) \propto \Gamma\left(2\left(10 + \frac{10y}{512}\right)^2, 10 + \frac{10y}{512}\right)(x) \times g(y),$$

with

$$g(y) = 10^2 g_{100,4}(y) + 5 g_{225,4}(y) + 10(e^{e^{-\frac{y^2}{10000}}} - 1) + 10^{-3},$$

g_{m,σ^2} being the gaussian pdf with mean m and variance σ^2 and $\Gamma(\alpha, \beta)$ being the gamma pdf with parameters α and β.

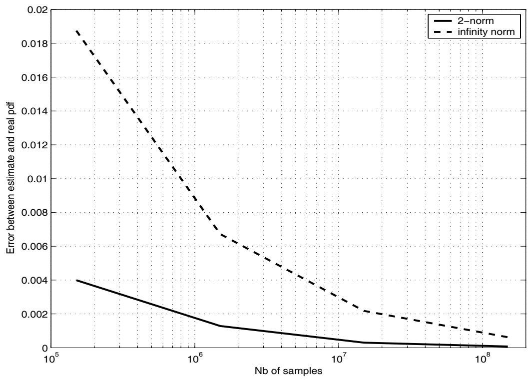

Fig. 3. Absolute error between initial density and its estimate

Since we are interested in the distribution of the energies, we also present the projection following the first dimension of the considered bidimensional densities. Figure 2-a is the contour representation of the initial pdf μ, and Figure 2-d the projection following the first dimension of this pdf. Figure 2-b represents the histogram deduced from the samples $\{(l_i, m_i)\}_{1 \leq i \leq N}$, and Figure 2-e its projection. Observe that the spikes and the continuum of the density both combine into pile-ups. We then apply the algorithms described in Section 2.

3.2. Results for pile-up correction algorithms

The density deduced from the algorithm of Table 2 is presented in Figure 2-c and its projection in Figure 2-f. Compared to the initial pdf, the estimate is quite sharp , even if the second gaussian cannot be easily distinguished from other pile-ups. Figure 3 presents a comparison between the ideal pdf μ and its estimate $\hat{\mu}$ deduced from our algorithm, the number of samples $\{(l_i, m_i)\}_{1 \leq i \leq N}$ varying. We choose two norms (the spectral radius and the infinity norm) as a criterion of sharpness. We compute in both cases the norm of $\mu - \hat{\mu}$. Figure 3 proves the efficiency of the method, and gives indication for the consistency of the estimate deduced from our algorithm.

Figure 4 presents the density estimate obtained by the the algorithm described in Table 1 and its marginal distribution. Consistency of the method following the number of points used in the numerical inversion of the Laplace transform should be investigated in future papers.

4. CONCLUSIONS AND PERSPECTIVES

In this paper we introduced a model based on two marked points processes that leads to a relation between two densities, one of them being of interest for the physicists in γ spectrometry. We exhibit algorithms to compute an estimate of this density in both case of continuous-time and discrete-time signals. This algorithm showed interesting results on generated densities. A problem associated to this should be to consider the case when the input Poisson process is not homogenous anymore. This question should be investigated in future papers.

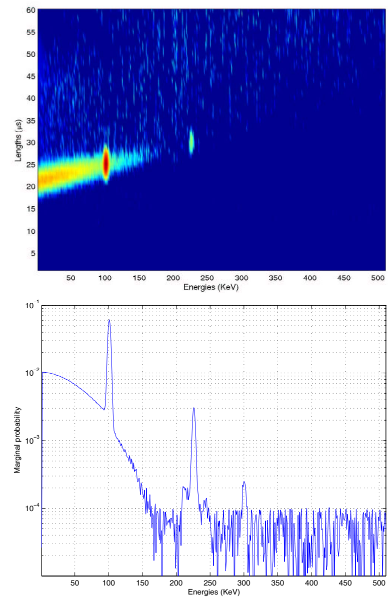

Fig. 4. Density estimate with the continuous version of the algorithm

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