## LMI Characterization for The Convex Hull of Trigonometric Curves and Applications

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Abstract-In this paper, we develop a new linear matrix inequality (LMI) technique, which is practical for solutions of the general trigonometric semi-infinite linear constraint (TSIC) of competitive orders. Based on the new full LMI characterization for the convex hull of a trigonometric curve, it is shown that the semi-infinite optimization problem involving TSIC can be solved by LMI optimization problem with additional variables of dimension just n, the order of the the trigonometric curve. Our solution method is very robust which allows us to address almost all practical filter design problems. Unlike most previous works involving several complex mathematical tools, our derivation arguments are based on simple results of the convex analysis and some formal elementary transforms. Furthermore, many filter/filterbank design problems can be reformulated as the optimization of linear/convex quadratic objectives over the trigonometric semiinfinite constraints (TSIC). Based on this reformulation, these problems can be equivalently reduced to LMI optimization problems with the minimal size. Our examples of designing up to 1200-tap filters verifies the viability of our formulation.

#### I. INTRODUCTION

A trigonometric curve is the set

$$C_{a,b} := \{ (1, \cos t, \cos 2t, ..., \cos nt)^T : \\ \cos t \in [\cos a, \cos b] \subset [-1, 1] \} \subset R^{n+1},$$
(1)

and its polar is defined as

$$C_{a,b}^* = \{ u \in \mathbb{R}^{n+1} : \langle u, v \rangle \ge 0 \ \forall \ v \in C_{a,b} \}.$$
 (2)

The trigonometric semi-infinite linear constraint (TSIC) in variable  $x \in R^{\bar{n}}$ 

$$Ax + d \in C^*_{a,b}, \ A \in R^{(n+1) \times \bar{n}}, \ d \in R^{n+1}$$
 (3)

includes several interesting interpretations in signal processing as a particular case. For instance, the particular case

$$x \in C^*_{\pi,0} \tag{4}$$

means that  $x = (x_0, x_1, ..., x_n)$  is a positive real sequence:

$$H(e^{j\omega}) := \sum_{h=0}^{n} x_h \cos h\omega \ge 0 \,\,\forall \,\,\omega \in [0,\pi].$$
 (5)

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The constraint  $Ax + d \in C^*_{\pi,0}$  also arises in problems such as FIR energy compaction filer design, signal-adapted filter banks etc. (see e.g. [6], [8] and references therein).

More generally, peak-error constraints [1] for frequency responses of the linear phase filters are the particular cases of (3): given a passband  $[0, \omega_p] \subset [0, \pi/2]$  and stopband  $[\omega_s, \pi]$ , the peak-error constraints in the passband and stopband of a linear phase filer

$$H(z) = z^{-n} [x_0 + \frac{1}{2} \sum_{i=1}^n x_i (z^i + z^{-i})]$$
(6)

which are expressed as

$$\begin{aligned} |H(e^{j\omega}) - e^{-jn\omega}| &< \delta_p \quad \forall \omega \in [0, \omega_p] \\ |H(e^{j\omega})| &< \delta_s \quad \forall \omega \in [\omega_s, \pi] \end{aligned} \tag{7}$$

are indeed

$$x + (\delta_p - 1)e_1 \in C^*_{\omega_p,0}, \ -x + (\delta_p + 1)e_1 \in C^*_{\omega_p,0}, x + \delta_s e_1 \in C^*_{\pi,\omega_s}, -x + \delta_s e_1 \in C^*_{\pi,\omega_s},$$
(8)

respectively, where  $e_1 = (1, 0, ..0) \in \mathbb{R}^{n+1}$ .

The simplest and traditional treatment for the TSIC constraints (3) is just to replace it by a finite number of linear constraints:

$$\langle Ax + d, (1, \cos t_i, \cos 2t_i, \dots, \cos nt_i)^T \rangle \ge 0, \\ \cos t_i \in [\cos a, \cos b], \ i = 0, 1, 2, \dots N.$$

$$(9)$$

Obviously, any feasible solution of these linear constraints is not guaranteed to be a feasible one of the TSIC constraints. As mentioned in [6] no matter as such a set of finite grid points  $\{t_i, i = 0, 1, ..., N\}$  is chosen dense on [b, a], linear constraints (9) cannot bring TSIC (3). On the other hand, in the state-space setting for the filter (5), the classical Kalman-Yakubovich-Popov lemma (see e.g. [9]) can reformulate the particular positive constraint (4) into (convex) linear matrix inequality (LMI) constraint involving additional Lyapunov symmetric matrix variable of dimension n(2n + 1).

Quite recently, LMI has been discovered as a powerful tool for handling general TSIC (3) as well. It has been established in [4] that TSIC (3) is characterized by LMI constraints involving two additional symmetric matrix variables of dimension (n+1)(n+2)/2 and n(n+1)/2, respectively. Usually the order 2n of the designed FIR filters is not small in practical applications to attain good frequency response. As a result, the dimension of the corresponding LMI optimization problem may be very high (with more than several thousands of additional variables), preventing them from being efficiently and practically solvable by existing LMI solvers such as [10]. A much more efficient LMI technique specialized for handling the particular positive constraint (4) has been developed in [2]. A numerical result has been provided in [2] for order n = 600 (for such order n, as mentioned, the corresponding LMI formulation [4] for (4) requires an additional variable of dimension (n + 1)(n + 2)/2 = 180901). It has been also shown in [5], [3] that the general TSIC (3) can be equivalently transformed to the particular positive real constraint (4). However, as recognized in [5], [3], the trouble there is that the resulting LMI constraint is ill-posed even for very moderate order n due the ill-condition of the used transform matrix. It should be noted that the mentioned LMI based formulation of [4] with much large dimensional variables is still to handle with the case n = 50. An alternative linear programming based method to handle the general TSIC (3) has been developed in [11], which allows to solve it for the order n up to 100. However, the case of higher order is still persistent.

In this paper, a new LMI technique is developed in practical for solutions of the general TSIC (3) of much more competitive orders. Based on the new full LMI characterization for the convex hull of the trigonometric curve  $C_{a,b}$  defined by (1), which is also of independent interest, we show that the semi-infinite optimization problem involving TSIC (3) can be solved by the LMI optimization problem with additional variables of dimension just *n*. The solution of our method is very robust and allows us to address almost all practical filter design problems. Our derivation arguments are only based on simple results of the convex analysis [9] and some formal elementary transforms.

The paper is organized as follows. After Section 1, the LMI characterization for the convex hull of the trigonometric curve (1) is developed in Section 2. This result is applied in Section 3 for robust LMI based solutions of optimization problems involving TSIC (3). Their applications to filter design problems are discussed in Sections 4. Due to the space limitation, all the proofs and derivations are omitted in this paper.

The notations used in the paper are standard. By  $X \ge 0$  we mean a (symmetric) positive semi-definite matrix while  $\langle .,. \rangle$  stands for the inner product of matrices, i.e.  $\langle X, Y \rangle = \text{Trace}(XY) = \text{Trace}(YX)$  for matrices X, Y. It is almost trivial fact that the dimension of the space of all  $n \times n$ -symmetric matrices is n(n + 1)/2. For a given set  $C \subset \mathbb{R}^n$  its convex hull (cone hull, resp.) is denoted by conv(C) (cone(C), which is the smallest convex set (smallest cone, resp.) containing C. The polar set of C is the cone  $C^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \ge 0 \ \forall y \in C\}$ .

# II. TRIGONOMETRIC MARKOV-LUKACS THEOREM AND CONVEX HULL

In this section we develop LMI characterization for the convex hull of the trigonometric curve (1).

Define the k-th moment trigonometric matrix  $T_k$  of size

 $(k+1) \times (k+1)$  as the positive semi-definite one

$$\mathcal{T}_k(t) = \begin{bmatrix} 1 \\ \cos t \\ . \\ \cos kt \end{bmatrix} \begin{bmatrix} 1 & \cos t & .. & \cos kt \end{bmatrix}^T$$
(10)

$$= \begin{bmatrix} 1 & \cos t & \dots & \cos kt \\ \cos t & \frac{\cos 2t+1}{2} & \dots & \frac{\cos(k+1)t+\cos(k-1)t}{2} \\ \dots & \dots & \dots & \dots & \dots \\ \cos kt & \frac{\cos(k+1)t+\cos(k-1)t}{2} & \dots & \frac{\cos 2kt+1}{2} \end{bmatrix}.$$

and accordingly, the matrix  $T_k(y)$  is created from  $T_k(t)$  by the variable change

$$\cos ht \leftarrow y_h, \ h = 0, 1, 2, ...,$$
 (11)

i.e.

$$\mathcal{T}_{k}(y) = \begin{bmatrix} y_{0} & y_{1} & \dots & y_{k} \\ y_{1} & \frac{y_{2}+y_{0}}{2} & \dots & \frac{y_{k+1}+y_{k-1}}{2} \\ \dots & \dots & \dots & \dots \\ y_{k} & \frac{y_{k+1}+y_{k-1}}{2} & \dots & \frac{y_{2k}+y_{0}}{2} \end{bmatrix}.$$
 (12)

Define also

$$\mathcal{T}_{\ell k}(t) = \cos \ell t \mathcal{T}_k(t) \tag{13}$$

and accordingly  $\mathcal{T}_{\ell k}(y)$  is created from  $\mathcal{T}_{\ell k}(t)$  by the variable change (11). Naturally, let  $\mathcal{T}_{0k}(t)$  denote  $\mathcal{T}_{k}(t)$ . The role of the introduced trigonometric moment matrices for trigonometric polynomials is similar to that of moment matrices for algebraic polynomials in the Markov-Lucacs theorem [7] as shown in the following theorem, which is one of our main results in this section.

Theorem 1: Any trigonometric polynomial  $P(t) = \sum_{h=0}^{n} p_h \cos ht$  of degree not more than *n*, which is nonnegative on [and n admits the representation]

ative on  $[\cos a, \cos b]$  admits the representation

$$P(t) = \langle X, \mathcal{T}_{k}(t) \rangle + \langle Z, (\cos b + \cos a) \mathcal{T}_{1(k-1)}(t) \\ -\frac{1}{2} \mathcal{T}_{2(k-1)}(t) - (\frac{1}{2} + \cos a \cos b) \mathcal{T}_{k-1}(t) \rangle, \\ X \ge 0, \ Z \ge 0, \quad \text{for } n = 2k, \quad (14)$$
$$P(t) = \langle \cos b \ Z - \cos a \ X, \mathcal{T}_{k}(t) \rangle + \langle X - Z, \mathcal{T}_{1k}(t) \rangle,$$

$$L = 0$$

 $X \ge 0, \ Z \ge 0,$  for n = 2k + 1.(15)By comparison of terms with the same "power"  $\cos h\omega$  at both sides of (14), (15) one can easily obtain an equivalent new LMI constraint characterization for the trigonometric semi-infinite constraint  $P(t) \ge 0 \ \forall \cos t \in [\cos a, \cos b].$ Then, as a consequence of the above Theorem we can obtain a new version of the Kalman-Yakubovich-Popov lemma for a positive real sequence as well. Moreover, the size of matrices X and Z in representations (14), (15) are  $([n/2] + 1) \times ([n/2] + 1)$  and  $[n/2] \times [n/2]$ (for *n* even) or  $([n/2] + 1) \times ([n/2] + 1)$  (for *n* odd) vs.  $(n + 1) \times (n + 1)$  and  $n \times n$  of their counterparts in LMI formulation of [4], i.e. their size have been substantially reduced. However, such size is still high for practical applications and we will see later that such drawback is avoided through the LMI characterization for the convex hull of the set  $C_{a,b}$ , which is formulated shortly.

*Theorem 2:* The convex hull of the set  $C_{a,b}$  defined by (1) is fully characterized by LMIs:

$$\operatorname{conv} C_{a,b} = \{(y_0, y_1, y_2, ..., y_n) : \mathcal{T}_k(y) \ge 0, \\ \mathcal{F}_k^{a,b}(y) \ge 0, \ y_0 = 1\}, \text{ for } n = 2k$$
(16)  
$$\operatorname{conv} C_{a,b} = \{(y_0, y_1, y_2, ..., y_n) : \operatorname{cos} b\mathcal{T}_k(y) \ge \mathcal{T}_{1k}(y)\}$$

$$\geq \cos a \mathcal{T}_k(y) \ y_0 = 1 \} \text{ for } n = 2k + 1$$
 (17)

where

$$\mathcal{F}_{k}^{a,b}(y) = (\cos b + \cos a)\mathcal{T}_{1(k-1)}(y)$$
(18)  
$$-\frac{1}{2}\mathcal{T}_{2(k-1)}(y) - (\frac{1}{2} + \cos a \cos b)\mathcal{T}_{k-1}(y)$$

Consequently, the cone hull of  $convC_{a,b}$  is defined by

cone
$$C_{a,b} = \{(y_0, y_1, y_2, ..., y_n) : \mathcal{T}_k(y) \ge 0,$$
  
 $\mathcal{F}_k^{a,b}(y) \ge 0\}, \text{ for } n = 2k$  (19)

$$coneC_{a,b} = \{(y_0, y_1, y_2, ..., y_n) : \cos b\mathcal{T}_k(y) \ge \mathcal{T}_{1k}(y) \\ > \cos a\mathcal{T}_k(y)\} \text{ for } n = 2k+1.$$
(20)

 $\geq \cos a I_k(y)$  for n = 2k + 1. (20) It is obvious that  $C_{a,b}^* = (\operatorname{conv}(C_{a,b}))^*$  so Theorem 2 allows us to handle the general TSIC (3) by a LMI with the minimal number of variables as we see through the next section.

#### III. APPLICATION TO CONVEX QUADRATIC OBJECTIVE OPTIMIZATION

For simplicity of description consider the case n = 2k. It is clear from our developemnt that the case n = 2k+1 is approached similarly. For the peak-error constrained filter design problem (see section 4), we have to deal with the convex quadratic objective

$$\min_{x} x^{T}Qx + c^{T}x : A_{i}x + d_{i} \in C^{*}_{a_{i},b_{i}}, \ i = 1, 2, ..., m,$$
(21)

(Q > 0), which by Theorem 1 is actually a LMI optimization problem but of a high dimension.

We will use the short notation  $C_i$  to refer cone $C_{a_i,b_i}$ . Then the dual problem of (21) is

$$\max_{y^{(i)} \in C_{i}} \min_{x} [x^{T}Qx + c^{T}x - \sum_{i=1}^{m} (A_{i}x + d_{i})^{T}y^{(i)}] = \begin{cases}
\max_{y^{(i)}} -\sum_{i=1}^{m} y^{(i)}d_{i} - \\
\frac{1}{4}(c - \sum_{i=1}^{m} A_{i}y^{(i)})^{T}Q^{-1}(c - \sum_{i=1}^{m} A_{i}^{T}y^{(i)}) \\
(19) \text{ for } a_{i} \leftarrow a, \ b_{i} \leftarrow b, \ i = 1, 2, ..., m,
\end{cases}$$
(22)

which can be rewritten as the LMI optimization program

$$\min_{y^{(i)},\nu} \sum_{i=1}^{m} y^{(i)} d_i + \nu : \\
\begin{bmatrix}
\nu & c^T - \sum_{i=1}^{m} y^{(i)T} A_i \\
c - \sum_{i=1}^{m} A_i^T y^{(i)} & 4Q
\end{bmatrix} \ge 0, \quad (23)$$
(19) for  $a_i \leftarrow a, \ b_i \leftarrow b, \ i = 1, 2, ..., m,$ 

in the sense that

$$\max(22) = -\min(23)$$

The optimal solution  $x_*, y_*^{(i)}$  of (21) and (22) must satisfy the complementary condition

$$\sum_{i=1}^{m} (A_i x_* + d_i)^T y_*^{(i)} = 0, \qquad (24)$$

$$x_* = -\frac{1}{2}Q^{-1}(c - \sum_{i=1}^m A_i^T y_*^{(i)})$$
(25)

It looks like that  $x_*$  must satisfy the overdetermined conditions (24) and (25). However, it can be easily shown that (24)-(25) are necessary and sufficient (Kuhn-Tucker condition) for the optimality of  $y_*^{(i)}$ . Therefore the optimal solution  $x_*$  of (21) is directly retrived from the optimal solution  $y_*^{(i)}$  of (22) by formula (25). One can see that the total number of scalar variables in (22) is mn vs their counterpart mn(n + 1)/2 + n in the corresponding LMI reformulation of (21).

#### IV. FIR FILTER DESIGN

Generally, a peak-constrained least-squares errors (PCLS) low-pass filter design problem can be formulated as follows: given a passband  $[0, \omega_p]$  and stopband  $[\omega_s, \pi]$ , we wish to design a linear phase filter such that the weighted-square error under the peak-error constraints (7) is minimized:

$$\min_{x} W_{p} \int_{0}^{\omega_{p}} |H(e^{j\omega}) - e^{-jn\omega}|^{2} d\omega \qquad (26)$$

$$+ W_{s} \int_{\omega_{s}}^{\pi} |H(e^{j\omega})|^{2} d\omega \quad \text{s.t.} \quad (7).$$

Clearly, the objective function in (26) is convex quadratic function  $x^TQx - q^Tx + r$  in the filter coefficients  $x = (x_0, x_2, ..., x_n)$ , where

$$Q = W_p \int_0^{\omega_p} \mathcal{T}_n(t) dt + W_s \int_{\omega_s}^{\pi} \mathcal{T}_n(t) dt, \qquad (27)$$

$$q = 2W_p \int_0^{\omega_p} [1 \cos t \dots \cos nt]^T dt, \quad (28)$$

$$r = W_p \omega_p. \tag{29}$$

The equivalent form of (26) is the following particular case of (21):

$$\min_{x=(x_0,x_1,\dots,x_n)^T} x^T Q x - q^T x + r : (8)$$
(30)

As mentioned above, the problem (30) can be solved by its dual (23), and after that its solution is calculated by (25). In our simulation, the problem (30) is considered with different data given in Table 1. The value of  $W_p$  and  $W_s$  is set to 2 and 2000, respectively.

Figure 1 show the frequency responses of the designed 401-tap filters. The correspnding optimal value of (23) is equal to 0.06, and the LHS of (24) is about 2.1538e - 009. The coefficients of the designed filter is calculated by formula (25). As described in Table 1, the passband is set

Filter length	$k_p$	$k_s$	$\frac{1+\delta_p}{1-\delta_p}$	$\delta_s$
(2n+1)			- • p	
401	0.03	0.04	1.0 DB	-80 DB
1201	0.1	0.105	1.0 DB	-100 DB
TABLE I				

Data of problem (30) in the simulation examples with  $\omega_p = 2\pi k_p \text{ and } \omega_s = 2\pi k_s$ 

to  $\omega_p = 2\pi 0.03$ , and the stopband is equal to  $\omega_s = 2\pi 0.04$ . We can see from Figure 1 that the filter frequency response in the passband is perfectly equal to 0db, and its value in the stopband is less than -80db. This means that the result of the designed filter strongly satisfies the peak constraints. In practice, we may like to design a FIR filter has the size of transition band, the ripples in passband and the frequency response of stopband as small as possible. As a result, we increase the order of filter to satisfy the designed constraints, but the tradeoff is the complexity of computation. This tradeoff in our method is mitigated greatly because the number of variables in our proposed method is equal to the number of coefficients of our designed filter. Similarly, Figure 2 presents the the frequency responses of the 1201tap filter. The corresponding optimal value of (23) is equal to 0.2, and the LHS of (24) is equal to 5.6533e - 009. In this case, the values of  $\omega_p$  and  $\omega_s$  are set to 0.1 and 0.105, respectively. Therefore, the shape of transition band is stretched vertically.

### V. CONCLUDING REMARKS

Certainly, LMI optimization provides a very useful tool for filter/filter-bank design problems. Until now, the application power of LMI is limited due to the artificial high dimension of LMI formulations for such problems. In this paper, we have developed a LMI formulation of moderate size for such problems that makes LMI not only useful



Fig. 1. Frequency response of the linear phase 401-tap FIR filter



Fig. 2. Frequency response of the 1201-tap linear phase FIR filter.

but powerfull tool as well. More advanced applications of our results in problems such as a design of biorthogonal cosine-modulated filter bank with about M = 32 channels and with length about M = 512 for ADSL (Asymmetric Digital Subscriber Line) in highbandwidth communication, a design of attenna pattern are under way. Our approach also works for multidimensional filter design as we will address it in an another work.

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