

ON CONFIDENCE INTERVALS FOR THE COHERENCE FUNCTION

Abdelhak M. Zoubir

Signal Processing Group
Darmstadt University of Technology
Merckstr. 25, D-64283 Darmstadt, Germany
e-mail: zoubir@ieee.org

ABSTRACT

We address the problem of confidence interval estimation for the coherence function. We revisit a recently proposed approach to calculate confidence intervals from the asymptotic distribution function of the sample coherence function. We then propose a bootstrap based approach and compare the level of confidence obtained by the two methods as well as a well-established method proposed by Enochson and Goodman. Extensive simulations have shown that the bootstrap approach is more accurate, in particular for a large coherence and non-Gaussian data.

1. PRELIMINARIES

Consider the real, 2 vector-valued stationary process $(X(t), Y(t))$ for $t = 0, \pm 1, \dots$. Assume that the process possesses the second order spectral densities $C_{XX}(\omega)$, $C_{YX}(\omega)$ and $C_{YY}(\omega)$ for $-\infty < \omega < \infty$. The coherence function, also called the magnitude-squared coherence function, of $Y(t)$ and $X(t)$, $t = 0, \pm 1, \dots$, at frequency ω is defined as

$$|R_{YX}(\omega)|^2 = \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)}, \quad -\infty < \omega < \infty \quad (1)$$

For simplicity we shall refer to $|R_{YX}(\omega)|^2$ as the coherence. It measures the extent to which $Y(t)$ is determinable from $X(t)$, $t = 0, \pm 1, \dots$, at frequency ω by linear time-invariant operations and is bounded between 0 and 1.

If $Y(t)$ is the output of a linear time-invariant filter plus noise, as depicted in Figure 1, then the spectral density of the noise process $\mathcal{E}(t)$ is given by $C_{\mathcal{E}\mathcal{E}}(\omega) = (1 - |R_{YX}(\omega)|^2)C_{YY}(\omega)$.

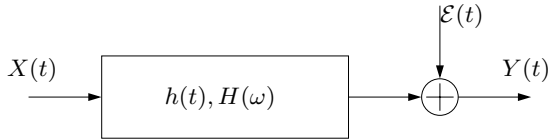


Fig. 1. Linear time-invariant filtering plus noise.

2. ESTIMATION OF THE COHERENCE

We consider non-parametric estimation of the coherence. Assume that we are given independent data records $X(t, l)$ and $Y(t, l)$ for $t = 0, \dots, T-1$ and $l = 1, \dots, n$. Define the finite Fourier

transforms of the data by $d_X(\omega, l) = \sum_{t=0}^{T-1} w(t/T)X(t, l)e^{-j\omega t}$ and $d_Y(\omega, l) = \sum_{t=0}^{T-1} w(t/T)Y(t, l)e^{-j\omega t}$ for $-\infty < \omega < \infty$, where $w(\tilde{t})$, $\tilde{t} \in \mathbb{R}$ is a window that is bounded, is of bounded variation and vanishes for $|\tilde{t}| > 1$. Let

$$\hat{C}_{XX}(\omega) = \frac{1}{nT} \sum_{l=1}^n |d_X(\omega, l)|^2, \quad \hat{C}_{YY}(\omega) = \frac{1}{nT} \sum_{l=1}^n |d_Y(\omega, l)|^2$$

and

$$\hat{C}_{YX}(\omega) = \frac{1}{nT} \sum_{l=1}^n d_Y(\omega, l) \bar{d}_X(\omega, l)$$

be estimates of $C_{XX}(\omega)$, $C_{YY}(\omega)$ and $C_{YX}(\omega)$, $-\infty < \omega < \infty$, respectively, obtained by averaging n periodograms, where we denoted by $\bar{d}_X(\omega, l)$ the complex conjugate of $d_X(\omega, l)$. An estimate of the coherence is obtained by

$$|\hat{R}_{YX}(\omega)|^2 = \frac{|\hat{C}_{YX}(\omega)|^2}{\hat{C}_{XX}(\omega)\hat{C}_{YY}(\omega)}. \quad (2)$$

Under regularity conditions, which include strict stationarity of the vector process $(X(t), Y(t))$, existence of all moments and absolute summability of all k -th order cumulant functions for all $k = 2, 3, \dots$, the asymptotic probability density function (pdf) of $|\hat{R}_{YX}(\omega)|^2$ (omitting frequency ω) is given by

$$(1 - |R_{YX}|^2)^n {}_2F_1(n, n; 1; |R_{YX}|^2 |\hat{R}_{YX}|^2) \times (n-1)(1 - |\hat{R}_{YX}|^2)^{n-2} \quad (3)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function, defined by

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!},$$

where $\Gamma(n+1) = n!$ and $(a)_n = a(a+1)(a+2)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n \geq 0$, with $(a)_0 = 1$. The asymptotic cumulative distribution function (cdf) is given by

$$\left(\frac{1 - |R_{YX}|^2}{1 - |\hat{R}_{YX}|^2} \right)^n |\hat{R}_{YX}|^2 \times \sum_{k=0}^{n-2} \left(\frac{1 - |\hat{R}_{YX}|^2}{1 - |R_{YX}|^2 |\hat{R}_{YX}|^2} \right)^k {}_2F_1(-k, 1-n; 1; |R_{YX}|^2 |\hat{R}_{YX}|^2). \quad (4)$$

The asymptotic pdf was given by Goodman [1] (see also [2]). Note that the choice of another estimator such as the smoothed

periodogram [3] or Welch's spectral estimate [4] would lead to the same results with due consideration given to the number of degrees of freedom of the estimation, i.e., n . The pdf and cdf expressions are exact if the vector-valued process is assumed to be Gaussian distributed.

3. CONFIDENCE INTERVAL ESTIMATION

For a given $|\hat{R}_{YX}(\omega)|^2$ of $|R_{YX}(\omega)|^2$, we wish to determine a lower bound $R_L(|\hat{R}_{YX}(\omega)|^2)$ and an upper bound $R_U(|\hat{R}_{YX}(\omega)|^2)$ so that for a given α

$$P(R_L(|\hat{R}_{YX}(\omega)|^2) \leq |R_{YX}(\omega)|^2 \leq R_U(|\hat{R}_{YX}(\omega)|^2)) = 1 - \alpha.$$

The random interval $(R_L(|\hat{R}_{YX}(\omega)|^2), R_U(|\hat{R}_{YX}(\omega)|^2))$ is called the confidence interval and covers the parameter $|R_{YX}(\omega)|^2$ with probability $1 - \alpha$. Under full knowledge of the distribution of $|\hat{R}_{YX}(\omega)|^2$ the confidence bounds are exact, otherwise they are approximations.

Recently Wang and Tang [5] proposed an iterative method which enables computation of a library of confidence intervals. The authors, without making any assumption on the distribution of the vector process, claim that they produce exact confidence intervals for the coherence $|R_{YX}(\omega)|^2$ while in fact they use the asymptotic cdf given in Eq. (4). They also claim that their method produces more accurate confidence intervals than approximations, especially when the number of data segments is small.

The prefix *exact* in [5] suggests that exact confidence bounds are drawn, but the authors use the asymptotic distribution given in Eq. (4). Also, the method proposed by the authors uses a Taylor series expansion which makes the proposed confidence interval an approximate one.

It should be noted that confidence intervals based on the cdf in Eq. (4) have been given by Amos and Koopmans in 1963. Examples of these for 80% and 95% confidence levels can be found in [6].

Enochson and Goodman [7] have also shown that for $n > 20$ and $0.4 \leq |R_{YX}(\omega)|^2 \leq 0.95$

$$|\check{R}_{YX}(\omega)| = \tanh^{-1} |\hat{R}_{YX}(\omega)|$$

is approximately normally distributed with mean $E|\check{R}_{YX}(\omega)| = \tanh^{-1} |R_{YX}(\omega)| + 1/(2(n-1))$ and variance $\text{var}|\check{R}_{YX}(\omega)| = 1/(2(n-1))$. Using this approximation an $100(1 - \alpha)\%$ confidence interval for $|R_{YX}(\omega)|^2$ can be readily obtained, where the lower and upper confidence bounds are given respectively by

$$\begin{aligned} & \tanh \left\{ \tanh^{-1} |\hat{R}_{YX}(\omega)| - \phi_{\alpha/2} \frac{1}{\sqrt{2(n-1)}} - \frac{1}{2(n-1)} \right\} \\ & \tanh \left\{ \tanh^{-1} |\hat{R}_{YX}(\omega)| + \phi_{\alpha/2} \frac{1}{\sqrt{2(n-1)}} - \frac{1}{2(n-1)} \right\}, \end{aligned}$$

where $\phi_{\alpha/2}$ is the upper $\alpha/2$ cutoff point for the standard normal distribution.

The purpose of the paper is to assess the accuracy of the confidence bounds suggested by Wang and Tang [5] and to also propose a bootstrap based confidence interval estimation as an alternative. Further, we will compute confidence intervals found with the Gaussian approximation in [7] and compare the results with those obtained with the method suggested by Wang and Tang [5] and our proposed bootstrap method depicted in Tables 1 and 2.

We note that in [8] bootstrap methods for the estimation of the distribution of the difference of (statistically dependent) sample multiple coherences have been discussed (see also [9]). Here, we will restrict our study to the coherence and omit the multiple coherence. However, all the methods discussed in this paper can be easily extended to the multiple coherence, as a generalisation of the coherence.

Bootstrapping coherences can be performed in two ways. One approach would be to explore a linear regression as a result of the model given in Figure 1. Omitting the error term $o_{a.s.}(1)$, which tends to zero almost surely as $T \rightarrow \infty$ [3], we have

$$d_Y(\omega) = H(\omega)d_X(\omega) + d_\varepsilon(\omega).$$

We would first estimate $H(\omega)$ through $\hat{H}(\omega) = \hat{C}_{YX}(\omega)/\hat{C}_{XX}(\omega)$ and define residuals

$$d_\varepsilon(\omega) = d_Y(\omega) - \hat{H}(\omega)d_X(\omega).$$

Given n independent data records, we would perform two totally independent resampling operations. We would draw, with replacement $\{d_X^*(\omega, 1), \dots, d_X^*(\omega, n)\}$ from $\{d_X(\omega, 1), \dots, d_X(\omega, n)\}$ and a resample $\{d_\varepsilon^*(\omega, 1), \dots, d_\varepsilon^*(\omega, n)\}$ from the residual data $\{d_\varepsilon(\omega, 1), \dots, d_\varepsilon(\omega, n)\}$. Bootstrap data for $d_Y(\omega)$ is obtained via the regression

$$d_Y^*(\omega) = \hat{H}(\omega)d_X^*(\omega) + d_\varepsilon^*(\omega),$$

where the joint distribution of $\{(d_X^*(\omega, l), d_Y^*(\omega, l)), 1 \leq l \leq n\}$, conditional on the sample of the measurement data $\mathcal{X}(\omega) = \{(d_X(\omega, 1), d_Y(\omega, 1)), \dots, (d_X(\omega, n), d_Y(\omega, n))\}$ is the bootstrap estimate of the unconditional joint distribution of $\mathcal{X}(\omega)$.

The above approach would enable us to replicate estimates of $|\hat{R}_{YX}(\omega)|^2$ with the bootstrap and estimate confidence bounds. However, the approach relies on the model of Figure 1. In the absence of any model, we propose the approach of Table 1, which makes use of the input-output data only.

In Table 1, we used a variance stabilising transformation h in order to get more accurate confidence intervals [9]. Its estimation is given in Table 2. Note that the variance estimation in Table 2 can also be performed using the jackknife in **Step b)**. It should also be noted that in this particular case we could use the transformation \tanh^{-1} which is known to be variance stabilising and would reduce computations.

4. THE EXPERIMENT

We consider a linear time-invariant (LTI) system as depicted in Figure 1, modelled by a finite impulse response (FIR) filter of second order whose frequency transfer function is given by $H(\omega) = 1 - 0.5e^{-j\omega} + 1.5e^{-j2\omega}$. The LTI system is driven by white noise $X(t)$, $t = 0, \pm 1, \dots$, with spectral density $C_{XX}(\omega) = \sigma_X^2$. Independent white noise $\mathcal{E}(t)$, with spectral density σ_ε^2 , is added to the output of the system to generate $Y(t)$, $t = 0, \pm 1, \dots$ (See Figure 1). It can easily be shown that in this case the coherence $|R_{YX}(\omega)|^2$ is given by

$$|R_{YX}(\omega)|^2 = \frac{1}{1 + |H(\omega)|^{-2} \sigma_\varepsilon^2 / \sigma_X^2}.$$

We construct confidence intervals for various true values of the coherence $|R_{YX}(\omega)|^2$ given above as follows. We generate independently $X(t, l)$ and $\mathcal{E}(t, l)$, $t = 0, \dots, T-1$ for $l = 1, \dots, n$

Table 1. The Bootstrap procedure.

Step 0. Data Collection.	Conduct the experiment and calculate the frequency data $d_X(\omega, 1), \dots, d_X(\omega, n)$ and $d_Y(\omega, 1), \dots, d_Y(\omega, n)$.
Step 1. Resampling.	Using a pseudo random number generator, draw a random sample, $\mathcal{X}^*(\omega)$ (of the same size), with replacement, from $\mathcal{X}(\omega) = \{(d_X(\omega, 1), d_Y(\omega, 1)), \dots, (d_X(\omega, n), d_Y(\omega, n))\}$.
Step 2. Bootstrap Estimates.	From $\mathcal{X}^*(\omega)$, calculate $ \hat{R}_{YX}^*(\omega) ^2$, the bootstrap analogue of $ \hat{R}_{YX}(\omega) ^2$ given in Eq. (2), replacing the estimates by their bootstrap counterparts and form $h(\hat{R}_{YX}(\omega) ^2)$ and $h(\hat{R}_{YX}^*(\omega) ^2)$.
Step 3. Repetition.	Repeat Steps 1-2 a large number of times to obtain a total of N bootstrap statistics $ \hat{R}_{YX}^{*1}(\omega) ^2, \dots, \hat{R}_{YX}^{*N}(\omega) ^2$.
Step 4. Distribution Function Estimation.	Sort the variance stabilised bootstrap estimates in increasing order to obtain $h^{(1)}(\hat{R}_{YX}^*(\omega) ^2) \leq \dots \leq h^{(N)}(\hat{R}_{YX}^*(\omega) ^2)$ and approximate the density function of $h(\hat{R}_{YX}(\omega) ^2) - h(R_{YX}(\omega) ^2)$ by the density function of $h(\hat{R}_{YX}^*(\omega) ^2) - h(\hat{R}_{YX}(\omega) ^2)$.
Step 5. Confidence Bands Estimation.	For a desired $(1 - \alpha)100\%$ bootstrap confidence interval, find critical points of the bootstrap distribution of $h(\hat{R}_{YX}^*(\omega) ^2) - h(\hat{R}_{YX}(\omega) ^2)$, $h^{(q_1)}(\hat{R}_{YX}^*(\omega) ^2)$ and $h^{(q_2)}(\hat{R}_{YX}^*(\omega) ^2)$, say, where $q_1 = \lfloor (N + 1)\alpha/2 \rfloor$ and $q_2 = N - q_1 + 1$. The confidence interval for $ R_{YX}(\omega) ^2$ is obtained as $(h^{-1}(h(\hat{R}_{YX}(\omega) ^2) - h^{(q_2)}(\hat{R}_{YX}^*(\omega) ^2)), h^{-1}(h(\hat{R}_{YX}(\omega) ^2) - h^{(q_1)}(\hat{R}_{YX}^*(\omega) ^2)))$.

independent data stretches. We filter $X(t, l)$, add $\mathcal{E}(t, l)$ to obtain $Y(t, l)$, $t = 0, \dots, T - 1$ for $l = 1, \dots, n$. Then, we estimate the coherence according to Eq. (2). We compute confidence intervals using the three following approaches:

- Wang and Tang method [5],
- Enochson and Goodman normal approximation [7], and
- the proposed bootstrap approach of Table 1 and Table 2.

For the method in [5], we have produced a library of confidence bounds for the coherence. The generation of the library is tedious and requires extensive computations to cater for a large range of sample coherences, but this computation needs to be performed only once. Given an estimate of the coherence, $|\hat{R}_{YX}(\omega)|^2$, one would use this library to find the confidence interval bounds. The bootstrap approach and the approach in [7] compute confidence intervals based directly on the coherence estimate.

Unless otherwise stated, we will use the following parameters: $\alpha = 5\%$, $N = 399$, $T = 64$, $n = 20$, $N_1 = 100$, $N_2 = 25$, $\text{SNR} = 0$ dB. All results presented in this section are based on 500 Monte Carlo replications. Consider first the case where the input signal and the noise are Gaussian distributed. Figure 2 shows the 95% confidence bounds obtained with all three methods. The figure indicates that the bootstrap and the method in [5] give

Table 2. Variance stabilising transformation estimation.

Step a) Resampling.	Generate N_1 bootstrap samples $\mathcal{X}_i^*(\omega)$ from $\mathcal{X}(\omega)$ and for each calculate $ \hat{R}_{YX}^{*i}(\omega) ^2$, $i = 1, \dots, N_1$.
Step b) Variance Estimation.	Generate N_2 bootstrap samples from $\mathcal{X}_i^*(\omega)$, $i = 1, \dots, N_1$ and calculate $\hat{\sigma}_i^{*2}$, a bootstrap estimate of the variance of $ \hat{R}_{YX}^{*i}(\omega) ^2$, $i = 1, \dots, N_1$.
Step c) Variance Function Estimation.	Estimate the variance function $\zeta(R_{YX}(\omega) ^2)$ by smoothing the values of $\hat{\sigma}_i^{*2}$ against $ \hat{R}_{YX}^{*i}(\omega) ^2$, $i = 1, \dots, N_1$.
Step d) Variance Stabilising Transformation Estimation.	Estimate the variance stabilising transformation from

$$h(|R_{YX}(\omega)|^2) = \int^{|R_{YX}(\omega)|^2} \zeta(s)^{-1/2} ds.$$

similar results, except at low coherence values while the method in [7], as expected, breaks down for coherence values outside the range of validity of the approximation.

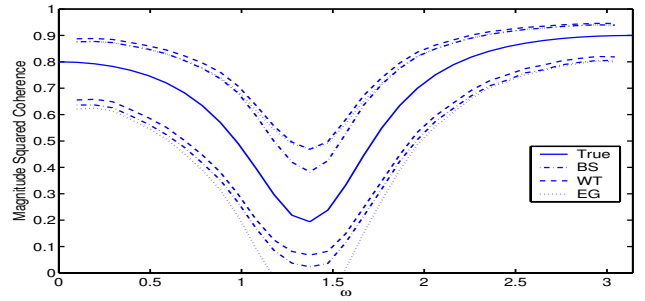


Fig. 2. True coherence (solid) and estimated 95% confidence bounds with the bootstrap (dash-dotted), Wang and Tang's method (dashed) and Enochson and Goodman's approach (dotted), for Gaussian signal and noise.

We also used the theoretical variance stabilising transformation \tanh^{-1} instead of the one given in Table 2. However, we divided the transformed statistic by the standard error, estimated using the bootstrap in order to reduce erratic behavior. This approach often leads to a better variance stabilisation than the bootstrap approach in Table 2. Using this type of variance stabilisation and for the setting above, we show in Figure 3 the actual upper and the lower tail coverages, i.e., $\hat{\alpha}/2$ as well as the confidence level $1 - \hat{\alpha}$ and the nominal values by solid lines. One can see that the bootstrap deviates more from the nominal value than the method in [5] for small coherence values.

We now consider non-Gaussian signals. For simplicity let us assume a constant coherence for all frequencies. Let the input signal be Laplace distributed and the noise be t_4 distributed. In Figure 4, we show the lower and upper tail coverages as well as the confidence level estimated by the three methods. It can clearly be seen that the bootstrap approach maintains the nominal levels better than the method suggested in [5]. Also, Enochson and Goodman's approximation performs well in comparison to the other two methods as the true coherence of 0.8 falls onto the range of valid

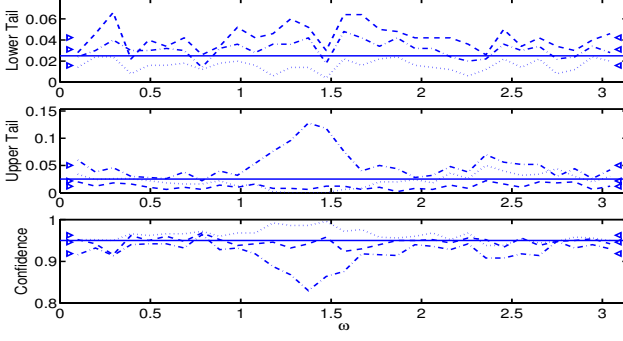


Fig. 3. Estimated lower and upper tail coverages and confidence level with the bootstrap (dash-dotted), Wang and Tang's method (dashed) and Enochson and Goodman's approach (dotted) for Gaussian signal and noise. Solid lines indicate nominal values.

Table 3. Lower and upper tails and confidence level for a nominal 95% confidence interval over all frequencies

	lower tail	confidence	upper tail
Bootstrap	0.034	0.937	0.030
Wang & Tang	0.061	0.923	0.016
Enochson & Goodman	0.030	0.940	0.030

approximation. The coverage values averaged over all frequencies are shown in Table 3.

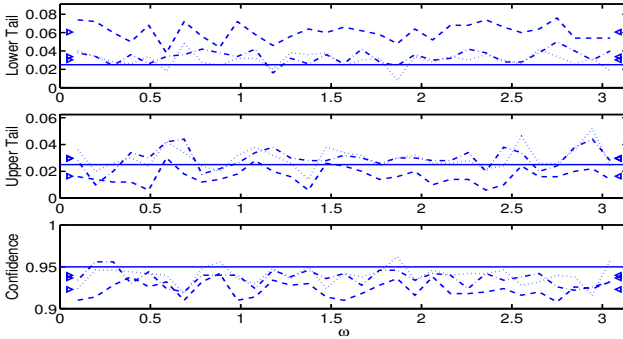


Fig. 4. Estimated lower and upper tail coverages and confidence level with the bootstrap (dash-dotted), Wang and Tang's method (dashed) and Enochson and Goodman's approach (dotted) for Laplace and t_4 distributed signal and noise, respectively. Solid lines indicate nominal values.

5. DISCUSSION

Extensive simulations have been performed with the three methods discussed in the paper. The results confirm that Wang and Tang's method is an approximation only. The method performs well when the distribution of the sample coherence is valid, i.e. when the signal and the noise are Gaussian or even in non-Gaussian settings when a large sample size n is available. As the sample size drops the method performs poorly. Also, the two other methods suffer in the latter case as the estimation of the coherence becomes critical,

especially for low values. We emphasize that the method proposed in [5] requires the use of a library. Although the library can be produced off-line, the complexity increases with the number of sample coherence points one wishes to consider. One can choose interpolation to overcome this problem, however.

The bootstrap method and Enochson and Goodman's approximation calculate confidence bounds directly from the sample coherence estimated from the data. The bootstrap can be computationally intensive if a variance stabilisation is estimated, but our experiments have shown that the theoretical \tanh^{-1} transformation with additional scaling by the standard error not only reduces computational complexity, but also leads to more accurate results.

6. CONCLUDING REMARKS

We have discussed confidence interval estimation for the coherence function. We have investigated three methods, of which one has recently been proposed and claimed to be exact. We have proposed a bootstrap based method, which, unlike the recently proposed method, does not require the generation of a library of confidence bounds. We have compared the results in view of confidence level and lower and upper tail coverages under different settings such as signal and noise distributions. Extensive simulations have shown that for a large coherence the bootstrap approach is more accurate in that it maintains the nominal confidence level, also in non-Gaussian environments.

7. ACKNOWLEDGEMENTS

The author wishes to thank Marco Moebus and Dr Ramon Bric for their help with the simulations.

8. REFERENCES

- [1] N. R. Goodman, "Statistical Analysis Based Upon a Certain Multivariate Complex Gaussian Distribution (An Introduction)," *Ann. Math. Statist.*, vol. 34, pp. 152–177, 1963.
- [2] G. C. Carter, "Coherence and Time Delay Estimation," *Proc. IEEE*, vol. 75, pp. 236–255, 1987.
- [3] D. R. Brillinger, *Time Series: Data Analysis and Theory*, Holden-Day, San Francisco, 1981.
- [4] P. D. Welch, "The Use of Fast Fourier Transform for Estimation of Power Spectra: A Method Based on Time Averaging over Short Modified Periodograms," *IEEE Trans. Audio Electroacoust.*, vol. AU-15, pp. 70–73, 1967.
- [5] S. Wang and M. Tang, "Exact Confidence Interval for Magnitude-Squared Coherence Estimates," *IEEE Signal Processing Letters*, vol. 11, pp. 326–329, 2004.
- [6] L. H. Koopmans, *The Spectral Analysis of Time Series*, Academic Press, New York, 1971.
- [7] L.D. Enochson and N.R. Goodman, "Gaussian Approximations to the Distribution of Sample Coherence," AFFDL-TR-65-57., Wright-Patterson Air Force Base, 1965.
- [8] A.M. Zoubir and B. Boashash, "The Bootstrap: Signal processing Applications," *IEEE Signal Processing Magazine*, vol. 15, pp. 56–76, 1998.
- [9] A. M. Zoubir and D. R. Iskander, *Bootstrap Techniques for Signal Processing*, Cambridge University Press, 2004.