A NEW LEAST-SQUARES-BASED MINIMUM VARIANCE SPECTRAL ESTIMATOR FAST ALGORITHM

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ABSTRACT

The traditional formulation of the minimum variance spectral estimator (MVSE) depends on the inverse of the autocorrelation matrix, which has a Toeplitz structure in the 1-D case. A fast computational algorithm exists that exploits this structure. This paper extends the class of fast MVSE algorithms to the case of a least-squares-based data-only formulation linked to the covariance case of linear prediction, which involves a near-to-Toeplitz matrix inverse. We show here that the inverse involves structures that yield fast computational formulations for the least-squares-based MVSE, in which the inverse has a special representation as sums of products of triangular Toeplitz matrices.

1. INTRODUCTION

The MVSE was originally introduced by Capon [1, Chap. 12] for use in multi-dimensional seismic array frequency-wavenumber analysis. A fast computational algorithm for evaluating the 1-D MVSE, which is a function of an inverse Toeplitz autocorrelation matrix, was discovered by Musicus in 1985 [2]. It exploited the structure of the Toeplitz inverse, which can be formulated in terms of triangular Toeplitz matrix products with matrix elements composed of 1-D autoregressive (AR) parameters. However, when the autocorrelation is unknown and only data samples are available, the 1-D MVSE can be formulated in least squares terms, which results in a MVSE expression involving an inverse of a matrix previously encountered in the covariance method of 1-D linear prediction. The matrix has a near-to-Toeplitz structure. In this paper, we show that the inverse of this near-to-Toeplitz matrix can also be formulated in terms of Toeplitz matrix products composed of parameters from the covariance linear prediction fast algorithm. These in turn can be substituted for the inverse matrix to obtain a fast computational algorithm for evaluating the 1-D least-squares-based covariance MVSE. The new fast algorithm not only inherits the estimation performance advantages of the least-squares-based MVSE over the autocorrelation-based MVSE, but also is a computationally efficient algorithm. Future papers will cover the 1-D least-squares-based modified covariance MVSE and the 2-D least-squares-based MVSE.

2. MVSE: KNOWN AUTOCORRELATION CASE

The MVSE is based on the concept of filtering the signal process $x[n]$ forward through the FIR filter

$$y^f[n] = \sum_{k=0}^{p} h[k]x[n-k] = \mathbf{x}_p^{\mathcal{T}}[n]\mathbf{h} \tag{1}$$

of order p and output $y^f[n]$, in which data vector $\mathbf{x}_p^{\mathcal{T}}[n] = \big(x[n] \;\; x[n-1] \;\; \cdots \;\; x[n-p]\big)$ is of dimension $p+1$, and likewise for filter vector $\mathbf{h}^{\mathcal{T}} = \big(h[0] \;\; h[1] \;\; \cdots \;\; h[p]\big)$. The statistical expectation of the filter output variance is simply

$$\rho = \mathcal{E}\{|y^f[n]|^2\} = \mathbf{h}^{\mathcal{H}}\mathcal{E}\{\mathbf{x}_p^*[n]\mathbf{x}_p^{\mathcal{T}}[n]\}\mathbf{h} = \mathbf{h}^{\mathcal{H}}\mathbf{R}_p\mathbf{h} \tag{2}$$

in which the $(p+1)\times(p+1)$ Toeplitz autocorrelation matrix is

$$\mathbf{R}_p = \begin{pmatrix} r[0] & \cdots & r^*[p] \\ \vdots & \ddots & \vdots \\ r[p] & \cdots & r[0] \end{pmatrix} \tag{3}$$

and $r[m] = \mathcal{E}\{x[n+m]x^*[n]\}$ are the autocorrelation sequence elements. We minimize the filter output variance subject to the constraint that, at a frequency f_0, the gain is unity, that is, $\mathbf{e}_p^{\mathcal{H}}(f_0)\mathbf{h} = 1$ in which the complex sinusoidal vector $\mathbf{e}_p^{\mathcal{T}}(f_0) = (1 \;\; \exp(j2\pi f_0 T) \;\; \cdots \;\; \exp(j2\pi f_0 pT))^{\mathcal{T}}$ and T is the sample interval. The result of the constrained minimization of the variance [1, page 353] is

$$\rho_{\mathrm{MV}} = \frac{1}{\mathbf{e}_p^{\mathcal{H}}(f_0)\mathbf{R}_p^{-1}\mathbf{e}_p(f_0)} \quad . \tag{4}$$

Scaling the variance by T yields units of power spectral density. Letting the frequency range over $-1/2T \leq f \leq$

$1/2T$, we arrive at the definition of the MVSE

$$P_{\mathrm{MV}}(f) = T\rho_{\mathrm{MV}} = \frac{T}{\mathbf{e}_p^{\mathcal{H}}(f)\mathbf{R}_p^{-1}\mathbf{e}_p(f)} \quad . \tag{5}$$

A fast computational algorithm, due to Musicus [2], is obtained by first noting that the inverse of the Toeplitz autocorrelation matrix can be expressed as the following difference of products of triangular Toeplitz matrices

$$\mathbf{R}_p^{-1} = \frac{1}{\rho_p}\mathbf{A}_p\mathbf{A}_p^{\mathcal{H}} - \frac{1}{\rho_p}\mathbf{B}_p\mathbf{B}_p^{\mathcal{H}} \tag{6}$$

in which the $(p+1)\times(p+1)$ triangular Toeplitz matrices

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_p[1] & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_p[p-1] & a_p[p-2] & \ddots & 1 & 0 \\ a_p[p] & a_p[p-1] & \cdots & a_p[1] & 1 \end{pmatrix}$$

$$\mathbf{B}_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_p^*[p] & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_p^*[2] & a_p^*[3] & \ddots & 0 & 0 \\ a_p^*[1] & a_p^*[2] & \cdots & a_p^*[p] & 0 \end{pmatrix}$$

are composed of AR parameters and white noise variance solutions from the Yule-Walker equation $\mathbf{R}_p\big(1 \;\; a_p[1] \;\; \ldots \;\; a_p[p]\big)^{\mathcal{T}} = \big(\rho_p \;\; 0 \;\; \ldots \;\; 0\big)^{\mathcal{T}}$. Substituting Eq. 6 for the inverse matrix into Eq. 5 yields

$$P_{\mathrm{MV}}(f) = \frac{T}{\sum_{k=-p}^{p}\psi_{\mathrm{MV}}[k]\exp(-j2\pi fkT)} \tag{7}$$

in which the complex conjugate symmetric coefficients $\psi_{\mathrm{MV}}[k] = \psi_{\mathrm{MV}}^*[-k]$ for $-p \le k \le -1$ are weighted correlations of the AR parameters

$$\psi_{\mathrm{MV}}[k] = \frac{1}{\rho_p}\sum_{i=0}^{p-k}(p+1-k-2i)a_p[k+i]a_p^*[i] \tag{8}$$

over the interval $0 \le k \le p$. Thus, FFTs can be used for a fast correlation computation and for evaluating the denominator expression of Eq. 7.

3. LS-MVSE: LEAST SQUARES DATA-ONLY CASE

The MVSE can also be formulated in terms of least squares minimization of the estimated variance when using only a finite data record, $x[n]$ for $1 \le n \le N$. The variance may then be estimated as

$$\begin{aligned} \hat{\rho}_p &= \frac{1}{(N-p)}\sum_{n=p+1}^{N}|y^f[n]|^2 \qquad (9) \\ &= \frac{1}{(N-p)}\mathbf{h}^{\mathcal{H}}\Big(\sum_{n=p+1}^{N}\mathbf{x}_p^*[n]\mathbf{x}_p^{\mathcal{T}}[n]\Big)\mathbf{h} \\ &= \frac{1}{(N-p)}\mathbf{h}^{\mathcal{H}}\big(\mathbf{X}_p^{\mathcal{H}}\mathbf{X}_p\big)\mathbf{h} \end{aligned}$$

in which the $(N-p)\times(p+1)$ rectangular Toeplitz data matrix is

$$\mathbf{X}_p = \begin{pmatrix} x[p+1] & \cdots & x[1] \\ \vdots & \ddots & \vdots \\ x[N-p] & \cdots & x[p+1] \\ \vdots & \ddots & \vdots \\ x[N] & \cdots & x[N-p] \end{pmatrix} \quad . \tag{10}$$

Minimizing the estimated variance subject to the same unit gain constraint as applied to the autocorrelation-based MVSE yields the following least-squares-based MVSE

$$P_{\mathrm{LSMV}}(f) = \frac{T}{\mathbf{e}_p^{\mathcal{H}}(f)\bar{\mathbf{R}}_p^{-1}\mathbf{e}_p(f)} \tag{11}$$

in which the product matrix $\bar{\mathbf{R}}_p = \mathbf{X}_p^{\mathcal{H}}\mathbf{X}_p$ does not have a Toeplitz structure, but does have a near-to-Toeplitz property similar as that described in [3]. This "nearness" can be exploited to yield the following sums of products of triangular Toeplitz matrices to represent the inverse [4]

$$\begin{aligned} \bar{\mathbf{R}}_p^{-1} &= \frac{1}{\rho_p^a}\mathbf{A}_p\mathbf{A}_p^{\mathcal{H}} - \frac{1}{\rho_p^b}\mathbf{B}_p\mathbf{B}_p^{\mathcal{H}} \qquad (12) \\ &\quad + \frac{1}{\rho_{p-1}^c}\mathbf{C}_{p-1}\mathbf{C}_{p-1}^{\mathcal{H}} - \frac{1}{\rho_{p-1}^d}\mathbf{D}_{p-1}\mathbf{D}_{p-1}^{\mathcal{H}} \end{aligned}$$

in which the $(p+1)\times(p+1)$ triangular Toeplitz matrices

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_p[1] & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_p[p-1] & a_p[p-2] & \cdots & 1 & 0 \\ a_p[p] & a_p[p-1] & \cdots & a_p[1] & 1 \end{pmatrix}$$

$$\mathbf{B}_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_p[p] & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_p[2] & b_p[3] & \ddots & 0 & 0 \\ b_p[1] & b_p[2] & \cdots & b_p[p] & 0 \end{pmatrix}$$

$$\mathbf{C}_{p-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{p-1}[0] & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{p-1}[p-2] & c_{p-1}[p-3] & \cdots & 0 & 0 \\ c_{p-1}[p-1] & c_{p-1}[p-2] & \cdots & c_{p-1}[0] & 0 \end{pmatrix}$$

$$\mathbf{D}_{p-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_{p-1}[0] & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{p-1}[p-2] & d_{p-1}[p-3] & \cdots & 0 & 0 \\ d_{p-1}[p-1] & d_{p-1}[p-2] & \cdots & d_{p-1}[0] & 0 \end{pmatrix}$$

are formed from the forward linear prediction parameters $a_p[m]$, forward linear prediction error variance ρ_p^a, backward linear prediction parameters $b_p[m]$, backward linear prediction error variance ρ_p^b, gain adjustment parameters $c_p[m]$ and $d_p[m]$, and scalar gain adjustment factors ρ_p^c and ρ_p^d. These are defined as matrix equations associated with the covariance linear prediction case

$$\begin{aligned} \bar{\mathbf{R}}_p \begin{pmatrix} 1 & a_p[1] & \cdots & a_p[p] \end{pmatrix}^{\mathcal{T}} &= \begin{pmatrix} \rho_p^a & 0 & \cdots & 0 \end{pmatrix}^{\mathcal{T}} \\ \bar{\mathbf{R}}_p \begin{pmatrix} b_p[p] & \cdots & b_p[1] & 1 \end{pmatrix}^{\mathcal{T}} &= \begin{pmatrix} 0 & \cdots & 0 & \rho_p^b \end{pmatrix}^{\mathcal{T}} \\ \bar{\mathbf{R}}_p \begin{pmatrix} c_p[0] & \cdots & c_p[p] \end{pmatrix}^{\mathcal{T}} &= \mathbf{x}_p^*[N] \\ \bar{\mathbf{R}}_p \begin{pmatrix} d_p[0] & \cdots & d_p[p] \end{pmatrix}^{\mathcal{T}} &= \mathbf{x}_p^*[p+1] \\ \rho_p^c &= 1 - \mathbf{x}_p^{\mathcal{T}}[N] \bar{\mathbf{R}}_p^{-1} \mathbf{x}_p^*[N] \\ \rho_p^d &= 1 - \mathbf{x}_p^{\mathcal{T}}[p+1] \bar{\mathbf{R}}_p^{-1} \mathbf{x}_p^*[p+1] \quad . \end{aligned}$$

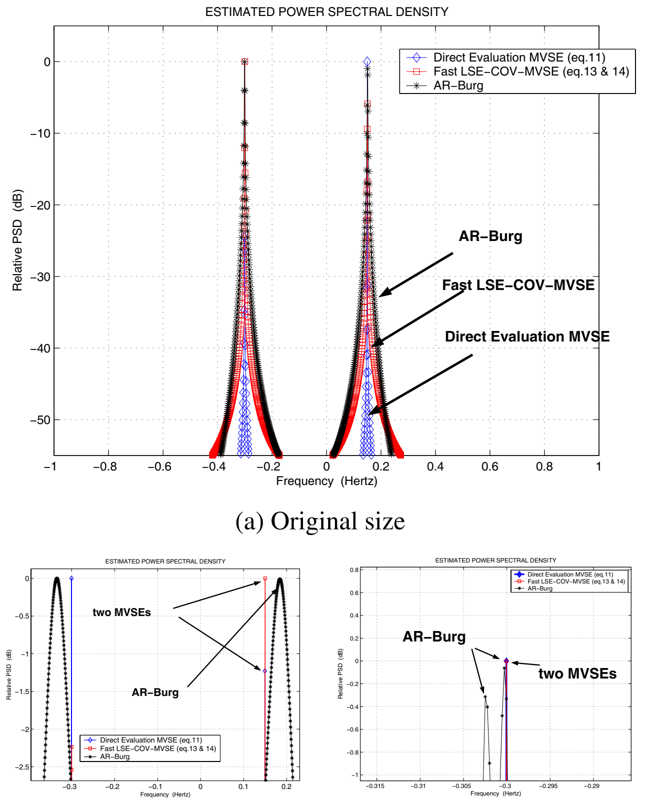
All of these parameters and factors are computed recursively as part of a normal execution of the fast computational solution to the least squares covariance case of linear prediction, as detailed in [1, Chap. 8]. Based on Eq. 12 substituted into Eq. 11, the complex conjugate symmetric $\bar{\psi}_{\text{MV}}[k]$ coefficients for the least-squares-based MVSE

$$P_{\text{LSMV}}(f) = \frac{T}{\sum_{k=-p}^{p} \bar{\psi}_{\text{MV}}[k] \exp(-j2\pi f k T)} \tag{13}$$

are

$$\begin{aligned} \bar{\psi}_{\text{MV}}[k] \quad &= \sum_{i=0}^{p-k} \Bigg(\frac{1}{\rho_p^a} (p+1-k-i) a_p[k+i] a_p^*[i] \\ &- \frac{1}{\rho_p^b}\, i\, b_p[i] b_p^*[k+i] \\ &+ \frac{1}{\rho_{p-1}^c} (p-k-i) c_{p-1}[k+i] c_{p-1}^*[i] \\ &- \frac{1}{\rho_{p-1}^d} (p-k-i) d_{p-1}[k+i] d_{p-1}^*[i] \Bigg) \end{aligned} \tag{14}$$

for $0 \leq k \leq p$ and where $\bar{\psi}_{\text{MV}}[k] = \bar{\psi}_{\text{MV}}^*[-k]$ for $-p \leq k \leq -1$. Note: (a) $a_p[0] = b_p[0] = 1$ and $c_{p-1}[p] = d_{p-1}[p] = 0$ by definition; (b) $c_{p-1}[0] \neq 0$ and $d_{p-1}[0] \neq 0$; (c) when calculating the linear prediction parameters and the gain adjustment parameters, the maximum order p_{max} must satisfy $p_{max} < (N-1)/2$, otherwise $\bar{\mathbf{R}}_p$ is not invertible by Eq. 12.



(a) Original size

(b) Unbiased LS-MVSE (c) LS-MVSE has no line splitting

Fig. 1. Power spectral estimate of a finite complex point test sequence for $p = 5$ and $f = 2$ Hz. (Notation: Blue diamond and red square lines are corresponding to the direct evaluation method and the fast algorithm of the LS-MVSE, respectively, compared with the Burg algorithm (black star line). Data samples in figure(a) and (b) are 200, in Figure(c) 20.)

The new algorithm requires $20\frac{1}{2}p^2 - (9N - 4\frac{1}{2})p + 11N$ multiply operations and $8\frac{1}{2}p^2 - (9N + 8\frac{1}{2})p + 13N$ add operations to calculate the set of $\bar{\psi}_{\text{MV}}[k]$, $0 \leq k \leq p$, coefficients. An FFT in Eq. 13 is then used to evaluate the denominator of the LS-MVSE over a range of frequencies. It only requires $5p + 12N$ memory locations to save all of the parameters. This is more efficient then direct evaluation of the original MVSE function Eq. 11, which requires, for each frequency, $\frac{2}{3}p^3 + \frac{5}{3}p^2 + (N+7)p + N$ multiply operations, $\frac{2}{3}p^3 + 2p^2 + (N + \frac{4}{3})p + N$ add operations and $p^2 + (4+N)p + N$ memory locations to store parameters.

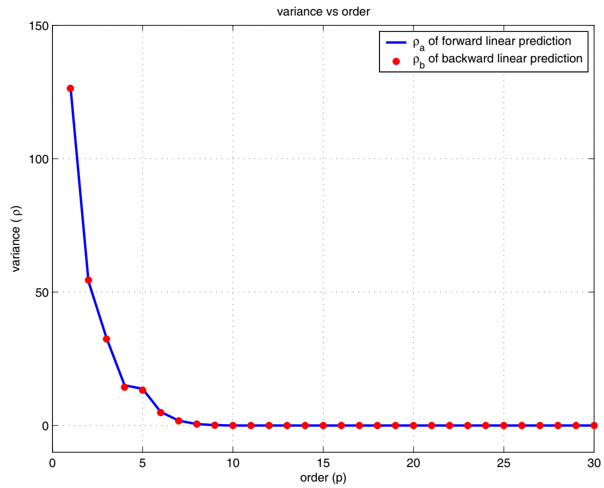


Fig. 2. Output variances, ρ_a and ρ_b, of forward and backward linear prediction decrease as order p increases.

4. SIMULATION RESULTS AND DISCUSSION

Because the Burg AR spectral estimate is one of the earliest and most frequently used spectral algorithms, we compared it with our new fast algorithm of the least-squares-based MVSE. Fig. 1 illustrates the MVSE based on both algorithms, direct evaluation of Eq. 11 and $\bar{\mathbf{R}}_p = \mathbf{X}^{\mathcal{H}}\mathbf{X}$, and the new fast algorithm of MVSE. The spectra are order 5 estimates produced from 200 complex samples of a process consisting of two sinusoids, of magnitude 100 units and frequencies f_1 = -0.3 Hz and f_2 = 0.15 Hz, with additive noise of variance 0.01 ($SNR = 60$ dB). The sampling rate f is 2 Hz. Fig. 1(a) shows that the new fast algorithm achieves comparable performance as the other two algorithms. It also inherits the advantage of sharper peaks from MVSE, especially when the data record is short. Burg's peaks are wider than those produced by the two MVSE methods. Moreover, increasing the number of the psd values in the vicinity of the peaks to zoom in on the peaks, one can see that the two biased peaks in the Burg spectral estimate are not exactly at -0.3 Hz and 0.15 Hz in Fig. 1(b). However, the two peaks of the two MVSE algorithms stay right on the two correct frequencies. The new fast algorithm inherits the unbiased spectral estimation property from the MVSE. As the data record decreased to 20 samples, Fig. 1(c) illustrates that the spectrum of the new fast algorithm of the least-squares-based MVSE does not exhibit the spectral line splitting ([1, Chap.8]) phenomenon of the Burg algorithm.

The new fast algorithm is not only more computationally efficient than the direct evaluation of the least-squares-based MVSE, it is also able to save all of the parameters of intermediate orders when it recursively calculates the parameters from order zero to order p. The output variances ρ_a and ρ_b will decrease to very small values as the order increases, as illustrated in Fig. 2 for a 128-complex point radar data record, sampling rate f = 100 Hz. By providing all intermediate order parameter values, this allows us to choose the optimal order. It is especially appropriate for applications in which the appropriate order is not known a priori. This allows us to choose the optimal order p_{opt} by making a plot like Fig. 2 without additional computations. Once the optimal order is selected, the saved parameters can then be used to compute Eq. 13 for this order. It is also easy to change the order without re-computing the set of required parameters. Therefore the new algorithm of the least-squares-based MVSE is computationally efficient and order selectable.

5. CONCLUSION

A new fast algorithm for solving the least-squares-based MVSE has been introduced in this paper. It has been shown that there is significant reduction of the computation complexity over the direct evaluation computation method of the least-squares-based MVSE. Computations proportional to p^2, and memory storage proportional to p are required, versus p^3 computations and p^2 storage for the direct solution method. The new fast algorithm saves all intermediate order parameters so that one can choose the optimal order and also can change the order without recalculating the parameters. The new fast MVSE algorithm inherits the advantages of the least-squares-based spectral estimation and is appropriate for applications in which the autocorrelation is unknown and only finite data samples are available.

6. REFERENCES

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