ROBUST FREQUENCY ESTIMATION BASED ON TRIMMED CORRELATION IN IMPULSIVE ENVIRONMENTS

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ABSTRACT

The MUSIC method represents a class of super-resolution methods for frequency estimation. However, it has poor performance in impulsive noise environments due to the presence of outliers. A more robust method called trimmed correlation based-MUSIC (TR-MUSIC) method is proposed in this paper. Through a trimming operation, outliers in the samples participating in the correlation calculation are discarded. The amount of trimming is determined by the Mahalanobis distance in which robust estimates of location and scale are utilized. Frequency estimation results from the eigendecomposition of the trimmed correlation matrix. Corroborating simulations are presented to show the robustness and performance improvement of the proposed method.

1. INTRODUCTION

Frequency estimation of sinusoidal signals embedded in white noise is frequently encountered in many applications such as radar, sonar and speech processing. It has been a classical problem in signal processing for many years. Among the methods proposed to address this problem, subspace based methods such as MUSIC and its variants, ESPRIT , and minimum norm method, have been extensively used due to their super-resolution capability. These approaches give very accurate frequency estimates in non-impulsive white noise case.

In some circumstances, however, noise may exhibit impulsive characteristics. For example, non-Gaussian α-stable noise has heavier tails than Gaussian noise and does not even possess finite variance. Impulsive noise occurs quite often in some practical situations, examples of which include radar clutter, underwater acoustics, and seismological measurements. The outliers appearing in the noise significantly distort the original signal, make the conventional autocorrelation estimates biased, and consequently degrade the performance of the aforementioned subspace based methods. Therefore, robust frequency estimators need to be developed to properly handle impulsive noise environments.

Several robust methods for either frequency or spectral estimation have been reported in the literature [1, 2, 3, 4]. In this paper, a new frequency estimation method which is based on MUSIC, but is more robust to impulsive noise, is proposed. In order to obtain a cleaned version of the data, a trimming operation is applied to the two vectors contributing to the correlation calculation. The resulting correlation estimate is referred to as the *trimmed correlation*. The MUSIC pseudospectrum is computed from the trimmed correlation matrix. The method is therefore called trimmed correlation based-MUSIC (TR-MUSIC). In this method, no other specific assumptions on noise are made except that the noise samples are i.i.d., which is a necessity for the MUSIC method. Another good feature is that the estimated trimmed-autocorrelation matrix retains the hermitian symmetric structure, thereby guarantees that eigenvalues of the trimmed correlation matrix are real. Also, it can be applied to real signals as well as complex signals. Simulation studies show that the proposed method performs very well even in very impulsive noise environments where the conventional MUSIC method fails to locate the correct frequencies. A comparison with another robust method [4] demonstrates the superior performance of the proposed method.

2. SIGNAL MODEL AND THE MUSIC METHOD

In this paper, the signals of interest are complex (real) sinusoids corrupted by additive noise. In general, a complex harmonic model can be described by

$$x(n) = \sum_{i=1}^{p} A_i \exp[j(2\pi f_i n + \phi_i)] + w(n), \quad n = 0, \dots, N-1 \tag{1}$$

where p is the total number of signal sources and is assumed to be known here, N is the total length of the observed data, f_i is the frequency of interest from the ith source, ϕ_i is the phase and is assumed to be uniformly distributed in $[0, 2\pi)$, A_i is the unknown amplitude, and $w(n)$ is the zero mean additive noise which is impulsive in nature. In addition, the noise samples are assumed to be i.i.d. as in the non-impulsive white noise case. Our aim is to find the p frequencies in (1).

The MUSIC spectrum is calculated as:

$$S_{\text{MUSIC}}(f) = \frac{1}{\sum_{i=p+1}^{M} |\mathbf{s}^H(f)\mathbf{v}_i|}, \quad (2)$$

where M is the dimension of the correlation matrix, $\mathbf{s}(f) = [1, e^{-j2\pi f}, \ldots, e^{-j2\pi f(M-1)}]^T$ and $\mathbf{v}_i$ is the eigenvector corresponding to the ith eigenvalue of the autocorrelation matrix. Note that eigenvalues are sorted in decreasing order since only the noise-subspace eigenvectors corresponding to $M-p$ smallest eigenvalues are of interest. Since the vector $\mathbf{s}(f)$ is orthogonal to $\mathbf{v}_i,\ i = p+1, \ldots, M$ at $f = f_i,\ i = 1, \ldots, p$, the summation in the denominator of (2) is zero. Consequently, the MUSIC method estimates the frequencies by picking the p frequencies where $S_{\text{MUSIC}}(f)$ attains peaks.

In conventional MUSIC method, autocorrelation matrix is estimated by sample correlation, which is quite susceptible to impulsive noise. Hence, the eigen-structure of the autocorrelation matrix is dramatically altered. Due to the large and possibly infinite variance of the noise, the eigenvalues in the signal subspace are indistinguishable from those in the noise subspace. This result has a detrimental effect on the conventional MUSIC method as in this case it is difficult to correctly select the $M-p$ eigenvectors in the noise subspace. Consequently, separation of signal and noise subspaces becomes a difficult problem in the impulsive noise scenario. Hence, in the following section, we propose a new method to improve the applicability of the MUSIC method in the impulsive noise case.

3. PROPOSED METHOD: TR-MUSIC

From robust statistics, we know that one approach to estimate the mean is using the α-trimmed mean [5]. Here, analogous to the α-trimmed mean, we introduce the α-trimmed correlation, or simply trimmed correlation. In the conventional sample correlation calculation, all samples participate in the calculation. However, by trimming those outlying samples, we may improve the correlation estimate. To elaborate this, we first discuss the real signal case. Extensions to complex signals are easily followed by some minor modifications.

In the real signal case, the sample correlation at time lag m is given by

$$\hat{r}_{xx}(m) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{n=0}^{N-m-1} x(n)x(n+m) & \text{if } m \geq 0, \\ \hat{r}_{xx}(-m) & \text{if } m < 0. \end{cases} \quad (3)$$

Without loss of generality, we examine the case $m \geq 0$. Denoting the vector $\mathbf{x}^b = [x(0), x(1), \ldots, x(N-m-1)]^T$ as the lagging vector and $\mathbf{x}^f = [x(m), x(m+1), \ldots, x(N-1)]^T$ as the leading vector, the sample autocorrelation is simply the inner product of these two vectors, i.e., $\hat{r}_{xx} = (\mathbf{x}^b)^T \cdot \mathbf{x}^f / N$. In the α-trimmed correlation, the parameter α is calculated as the proportion of outlying samples, which is given by $\alpha =$ (total # of outliers)$/N$. To appropriately trim the outliers, we first order the samples in the vector $\mathbf{x}^b$ and $\mathbf{x}^f$, thus obtaining two new ordered vectors

$$\mathbf{x}_o^b = [x_{(1)}^b, x_{(2)}^b, \ldots, x_{(N-m)}^b]^T \quad (4a)$$

$$\mathbf{x}_o^f = [x_{(1)}^f, x_{(2)}^f, \ldots, x_{(N-m)}^f]^T, \quad (4b)$$

where the samples are arranged in nondecreasing order. Once the percent of trimming α is determined, the number of samples trimmed is set as

$$t = \lfloor \alpha(N-m) \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the largest integer that is less than the argument. Then we trim the t smallest and largest samples in both $\mathbf{x}_o^b$ and $\mathbf{x}_o^f$ and the remaining samples are stored for further processing.

To determine the parameter α, a quantitative measure is needed to detect the outliers in the samples. One such measure is given by the *Mahalanobis distance* (MD) [6], which measures the distance of the data vector to the central mass of the whole set of data. In the case at hand, we have only single time series and can define the MD of the scalar random variable as

$$d(x(n), \mu_x) = \frac{[x(n) - \mu_x]^2}{\sigma_x^2}, \quad (5)$$

where μ_x is the mean and σ_x^2 is the variance. Samples with MD values larger than a preset threshold ξ are marked as outliers. Outlying samples usually have much larger MD than the rest. Hence, selection of the threshold ξ can be based on the computation of the MD for the whole data set $\{x(n)\}_{n=0}^{N-1}$.

Note that in practice, μ_x and σ_x^2 in the calculation of (5) are unknown and must be estimated from the data. Sample mean and sample variance could be used if there were only very few outliers. Utilizing this approach in the presence of a significant amount of outliers, however, makes the MD of outliers less than those of the normal samples. Hence, we adopt more robust estimates for the location and scale parameters. For μ_x, we replace it by the sample median

$$\hat{\mu}_x = \text{med}(x(0), x(1), \ldots, x(N-1)). \quad (6)$$

For σ_x^2, we replace it by the square of the median absolute deviation (MAD)

$$\hat{\sigma}_x^2 = \left(\text{med} \left\{ \left| x(n) - \underset{0 \leq n \leq N-1}{\text{med}} x(n) \right| \right\} \right)^2. \quad (7)$$

Since the autocorrelation involves temporal relationship between samples, the temporal information of the remaining samples after trimming must be retained. In addition, when a sample $x(n)$ in $\mathbf{x}_o^b$ is trimmed, its counterpart $x(n+m)$ in $\mathbf{x}_o^f$ should also be eliminated from correlation calculation and vice versa. However, the trimming procedures described above does not take this into account. In the following, we consider this problem by using set operations. We name such kind of procedure as *retrimming*.

Denote the set of the time indices of samples in the ordered lagging vector $\mathbf{x}_o^b$ as

$$\mathcal{I}^b = \{i_1, \ldots, i_t, i_{t+1}, \ldots, i_{N-m-t}, i_{N-m-t+1}, \ldots, i_{N-m}\} .$$

We also define $\mathcal{T}^b$ as the set of time indices corresponding to the samples trimmed from $\mathbf{x}_o^b$

$$\mathcal{T}^b = \{i_1, \ldots, i_t, i_{N-m-t+1}, \ldots, i_{N-m}\} .$$

Consequently, the set of the time indices of the surviving samples after trimming in $\mathbf{x}_o^b$ is given by

$$\mathcal{R}^b = \mathcal{I}^b - \mathcal{T}^b = \{i_{t+1}, i_{t+2}, \ldots, i_{N-m-t}\} .$$

Similarly, we can define the sets of the time indices for the total samples, trimmed samples, and surviving samples in $\mathbf{x}_o^f$ as $\mathcal{I}^f$, $\mathcal{T}^f$, and $\mathcal{R}^f$ respectively. Note that the time indices in the set $\mathcal{I}^f$ are those in $\mathcal{I}^b$ shifted by the autocorrelation lag m. Therefore, subtracting each element in $\mathcal{T}^f$ by m will map the time indices in $\mathcal{T}^f$ to those in $\mathcal{T}^b$. We can thus define a new index set

$$\bar{\mathcal{T}}^f = \{k - m : k \in \mathcal{T}^f\} ,$$

which is formed from the index set $\mathcal{T}^f$ with each element shifted by m. Since the trimming operation is performed on both $\mathbf{x}_o^b$ and $\mathbf{x}_o^f$, the complete set of trimmed samples must be taken into account. Consequently, the time index set of the remaining samples that will participate in the trimmed correlation calculation is given by

$$\mathcal{R} = \mathcal{I}^b - (\mathcal{T}^b \cup \bar{\mathcal{T}}^f) . \tag{8}$$

Finally, the trimmed autocorrelation is defined as

$$\hat{r}_{xx}^{\mathrm{tr}}(m) = \begin{cases} \dfrac{1}{N_{\mathcal{R}}} \displaystyle\sum_{n \in \mathcal{R}} x(n)x(n+m) & \text{if } m \geq 0, \\ \hat{r}_{xx}^{\mathrm{tr}}(-m) & \text{if } m < 0, \end{cases} \tag{9}$$

where $N_{\mathcal{R}} = |\mathcal{R}|$ is the cardinality of $\mathcal{R}$.

The trimmed autocorrelation retains the symmetry property since for negative lags, we only need to switch the role of the lagging vector $\mathbf{x}_o^b$ and the leading vector $\mathbf{x}_o^f$. This is a desired property for the autocorrelation since it results in a symmetric autocorrelation matrix $\hat{\mathbf{R}}_{xx}^{\mathrm{tr}}$.

In the α-trimmed correlation, trimming is performed based on the parameter α, which represents the percentage of outliers and is determined by computing the MD for the whole data. As an alternative approach, one may trim the leading and lagging vectors by directly computing the MD for each sample in the two vectors and discard those samples with MD greater than a preset threshold. Similar procedures can be carried out to retrim samples with temporal correspondence in the two vectors as done in the α-trimmed correlation. However, this approach is inefficient because it needs to evaluate the MD for each sample at each correlation lag. If we need to calculate correlation for large number of lags, the computational burden is formidable. In addition, we find through the simulations that trimming based on α-trimmed correlation yields better performance than trimming directly based on MD. Hence, we adopt α-trimmed correlation and utilize it to estimate the frequencies.

The trimmed correlation based MUSIC spectrum for real signal is formed on the basis of eigen-decomposition of $\hat{\mathbf{R}}_{xx}^{\mathrm{tr}}$, which can be readily computed as

$$S_{\mathrm{TR-MUSIC}}^{r}(f) = \frac{\mathbf{e}^T(f)\mathbf{e}(f)}{\displaystyle\sum_{i=2p+1}^{M} |\mathbf{e}^T(f)\mathbf{v}_i^{\mathrm{tr}}|} , \tag{10}$$

where $\mathbf{v}_i^{\mathrm{tr}}$ is the noise subspace eigenvector of the trimmed autocorrelation matrix $\hat{\mathbf{R}}_{xx}^{\mathrm{tr}}$ and $\mathbf{e}(f) = [1, \cos(2\pi f), \ldots, \cos(2\pi(M-1)f)]^T$. The frequencies are estimated by locating p peaks in $S_{\mathrm{TR-MUSIC}}(f)$ over the normalized frequency range $[0, 1/2]$ for real signals.

The proposed TR-MUSIC method can be easily extended to complex signals case. In that case, we can order the samples according to their magnitudes and also change the MD to reflect magnitude ordering. The rest of work is similar to the real signal case.

4. CASE STUDIES

We use simulations to demonstrate the results obtained from the TR-MUSIC method proposed in this paper. We show the performance improvement of the TR-MUSIC over the MUSIC and also compare it with the SIGN-MUSIC proposed in [4]. Since the complex signal case is similar to the real signal case and the SIGN-MUSIC is only applicable to real signals, we show a real signal example in the following.

We model the impulsive noise as α-stable noise in the presented simulations. In the simulation, both spectrum figures and normalized mean squared error (NMSE) figure are given for comparison. Each spectrum figure is generated by 30 overlaid independent realizations. The NMSE figure is plotted from the results of 1000 independent Monte Carlo runs. The total data length N is 1024. For the definition of signal-to-noise ratio (SNR) in α-stable noise case, we

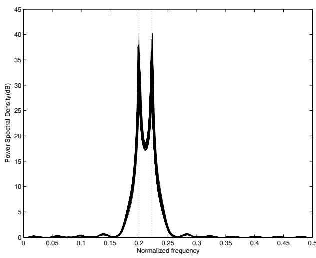

Fig. 1. TR-MUSIC spectrum

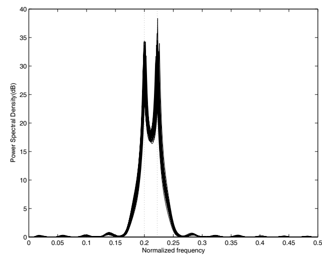

Fig. 2. SIGN-MUSIC spectrum

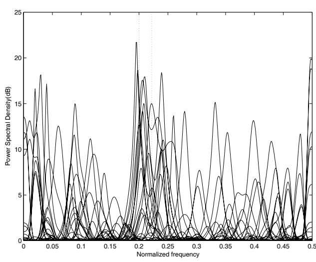

Fig. 3. MUSIC spectrum

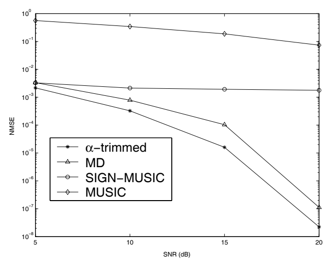

Fig. 4. NMSE

adopt the SNR definition in which the geometric power of α-stable noise is used

$$\mathrm{SNR} = \frac{P_s}{P_n} = \frac{A^2}{\frac{(C_g \gamma)^{2/\alpha'}}{C_g^2}} \,, \tag{11}$$

where A^2 is the signal power, γ is the dispersion of the α-stable noise, α' is the characteristic exponent and $C_g \approx 1.7811$ is the exponential of the Euler constant. To compare the performance of the TR-MUSIC with the MUSIC and the SIGN-MUSIC, we quantify the performance using the normalized mean squared error

$$\mathrm{NMSE} = \frac{1}{N_t} \sum_{i=1}^{N_t} \frac{\|\hat{\mathbf{f}}_i - \mathbf{f}_0\|^2}{\|\mathbf{f}_0\|^2} \,. \tag{12}$$

In the above expression, N_t is the number of trials, $\|\cdot\|$ is the norm of a vector, $\hat{\mathbf{f}}_i$ is the vector of frequency estimates for ith trail, and $\mathbf{f}_0$ is the true frequency vector.

The TR-MUSIC method is applied to a real signal consisting of two sinusoidal signals with frequencies $f_1 = 0.2$ and $f_2 = 0.222$ embedded in α-stable noise. The parameters of the α-stable noise are $\alpha' = 1.1$, $\beta = 0$ and $\mu = 0$. To better illustrate the point that α-trimmed correlation performs better than trimming based on MD only, we also plot the NMSE curve for trimming based on MD for comparison. Figures 1-3 show the spectrum estimated from the TR-MUSIC, SIGN-MUSIC, and MUSIC, respectively, for the SNR=5 dB. The threshold ξ is chosen to be 30. It can be seen from these figures that both TR-MUSIC and SIGN-MUSIC methods are capable of identifying the two frequencies in the presence of impulsive noise, while the conventional MUSIC completely fails. Also, from Fig. 4, we can see that both TR-MUSIC methods have smaller MSE than the SIGN-MUSIC method. Furthermore, the α-trimmed based TR-MUSIC performs slightly better than the MD-based TR-MUSIC.

5. CONCLUSION

Frequency estimation of sinusoidal signals in impulsive noise is addressed in this paper. Conventional subspace methods, like MUSIC, cannot guarantee to find the frequencies correctly in this scenario. A novel robust trimmed correlation based-MUSIC method is proposed for this purpose. In calculating the autocorrelation of the signal, trimming is performed on both lagging and leading vectors so that outliers are removed from the samples involved in the correlation estimates. The autocorrelation matrix is formed based on the trimmed correlation sequence and the frequencies are estimated from the eigendecomposition of the trimmed autocorrelation matrix. Simulations show that the TR-MUSIC performs better than the MUSIC and another robust method SIGN-MUSIC in impulsive noise environments.

6. REFERENCES

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