Discrete Fractional Fourier Transform Based on New Nearly Tridiagonal Commuting Matrices

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ABSTRACT

Based on discrete Hermite-Gaussian like functions, a discrete fractional Fourier transform (DFRFT) which provides sample approximations of the continuous fractional Fourier transform was defined and investigated recently. In this paper, we propose a new nearly tridiagonal matrix which commutes with the discrete Fourier transform (DFT) matrix. The eigenvectors of the new nearly tridiagonal matrix are shown to be better discrete Hermite-Gaussian like functions than those developed before. Furthermore, by appropriately combining two linearly independent matrices which both commute with the DFT matrix, we develop a method to obtain even better discrete Hermite-Gaussian like functions. Then, new versions of DFRFT produce their transform outputs more close to the samples of the continuous fractional Fourier transform, and their application is illustrated.

1. INTRODUCTION

The a^{th}-order continuous fractional Fourier transform (FRT) of *x(t)* is defined as [4]

$$X_a(u) = \int_{-\infty}^{+\infty} x(t) K_a(t,u)\,dt, \tag{1}$$

where the transform kernel $K_a(t,u)$ is given by

$$K_a(t,u) = \sqrt{1 - j\cot\alpha} \cdot e^{j\pi(t^2 \cot\alpha - 2tu\csc(\alpha) + u^2 \cot\alpha)}, \tag{2}$$

in which $\alpha = a\pi/2$. It is known that the transform kernel $K_a(t,u)$ can also be written as [4]

$$K_a(t,u) = \sum_{n=0}^{\infty} \exp(-jna\pi/2) \cdot \Psi_n(t)\Psi_n(u), \tag{3}$$

where

$$\Psi_n(t) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi} \cdot t) e^{-\pi \cdot t^2} \tag{4}$$

is the n^{th}-order Hermite-Gaussian function with H_n being the n^{th}-order Hermite polynomial.

The $N \times N$ DFT matrix **F** is defined by

$$\mathbf{F}_{kn} = (1/\sqrt{N}) \cdot e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k, n \le N-1. \tag{5}$$

In [5], Dickinson and Steiglitz introduced an $N \times N$ nearly tridiagonal matrix **S** whose nonzero entries are:

$$\begin{aligned} &\mathbf{S}_{n,n} = 2\cos\left(\frac{2\pi}{N} \cdot n\right), \quad 0 \le n \le (N-1) \\ &\mathbf{S}_{n,n+1} = \mathbf{S}_{n+1,n} = 1, \quad 0 \le n \le (N-2) \\ &\mathbf{S}_{N-1,0} = \mathbf{S}_{0,N-1} = 1. \end{aligned} \tag{6}$$

With **S** defined above, **S** commutes with **F**, i.e., **SF**=**FS**. Therefore, the DFT matrix **F** and the above matrix **S** share a common eigenvector set and we can find the eigenvectors of **F** from those of the matrix **S** [3].

Analogous to the spectral expansion of the continuous FRT kernel $K_a(t,u)$ in (3), and from the fact that the eigenvectors of **S** can be used as the discrete Hermite-Gaussian like functions, in [2], Pei et al. defined the a^{th}-order DFRFT matrix $\mathbf{F}_\mathbf{S}^a$ by

$$\mathbf{F}_\mathbf{S}^a = \mathbf{V}\mathbf{D}^a\mathbf{V}^T = \begin{cases} \sum_{k=0}^{N-1} e^{-j\frac{\pi}{2}ka} \mathbf{v}_k \mathbf{v}_k^T, & \text{for } N \text{ odd} \\ \sum_{k=0}^{N-2} e^{-j\frac{\pi}{2}ka} \mathbf{v}_k \mathbf{v}_k^T + e^{-j\frac{\pi}{2}Na} \mathbf{v}_N \mathbf{v}_N^T, & \text{for } N \text{ even} \end{cases} \tag{7}$$

where T denotes the matrix transpose, the matrix $\mathbf{V} = [\mathbf{v}_0 | \mathbf{v}_1 | \cdots | \mathbf{v}_{N-2} | \mathbf{v}_{N-1}]$ for odd N and $\mathbf{V} = [\mathbf{v}_0 | \mathbf{v}_1 | \cdots | \mathbf{v}_{N-2} | \mathbf{v}_N]$ for even *N*, **D** is a diagonal matrix with its diagonal entries corresponding to the eigenvalues for each column eigenvectors in **V**, and $\mathbf{v}_k$ is the k^{th}-order discrete Hermite-Gaussian like function with k zero-crossings and is obtained from the corresponding normalized eigenvector of **S**. The **S**-based DFRFT of **x** can be easily obtained by $\mathbf{y}_a = \mathbf{F}_\mathbf{S}^a \mathbf{x}$.

2. A NEW NEARLY TRIDIAGONAL COMMUTING MATRIX T

In [1], Grünbaum introduced an exactly tridiagonal matrix commuting with the centered discrete Fourier transform matrix of even size. Inspired by the work of Grünbaum, we propose in this section a novel nearly tridiagonal matrix which commutes with the ordinary DFT matrix of any size, even or odd. Moreover, we will demonstrate that its eigenvectors approximate samples of the continuous Hermite-Gaussian functions better than those of the **S** matrix. Therefore, we can intuitively expect better performance of defining its DFRFT using the new nearly tridiagonal matrix.

Let us define an $N\times N$ nearly tridiagonal matrix $\mathbf{T}$ whose nonzero entries are (note that the matrix indices are from 0 to N-1):

$$\mathbf{T}_{n,n} = \cos^2(\frac{n\pi}{N}), \quad 0 \le n \le (N-1)$$

$$\mathbf{T}_{n,n+1} = \mathbf{T}_{n+1,n} = \frac{\cos\frac{n\pi}{N}\cos\frac{(n+1)\pi}{N}}{2\cos(\pi/N)}, \quad 0 \le n \le (N-2) \tag{8}$$

$$\mathbf{T}_{N-1,0} = \mathbf{T}_{0,N-1} = 0.5.$$

Note that except for the two 0.5 entries at the upper-right and lower-left corners, $\mathbf{T}$ is tridiagonal, which is similar to the $\mathbf{S}$ matrix of (6). Thus we call them nearly tridiagonal. Since $\mathbf{T}$ is real and symmetric, $\mathbf{T}$ has real and orthogonal eigenvectors. Besides, $\mathbf{T}$ has the following important property for this paper.

Property 1: The $N\times N$ matrix $\mathbf{T}$ commutes with the $N\times N$ DFT matrix $\mathbf{F}$ defined in (5), i.e., $\mathbf{TF}=\mathbf{FT}$.

From *Property 1*, it can be seen that if $\mathbf{x}$ is the eigenvector of $\mathbf{T}$ corresponding to an eigenvalue of multiplicity 1, then $\mathbf{x}$ is also an eigenvector of $\mathbf{F}$. It can be shown that the entries of the eigenvectors of $\mathbf{T}$ are solutions of a discrete version of the defining second-order differential equation of the continuous Hermite-Gaussian functions [3]. Therefore, the eigenvectors of $\mathbf{T}$ are discrete Hermite-Gaussian like functions. To motivate our further discussions, we perform some computer experiments to show that the eigenvectors of $\mathbf{T}$ approximate samples of continuous Hermite-Gaussian functions better than those of $\mathbf{S}$.

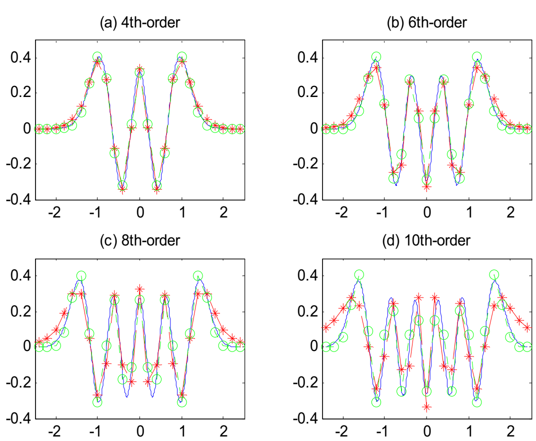

Fig. 1. The continuous Hermite-Gaussian functions (solid line), the discrete Hermite-Gaussian like functions based on $\mathbf{S}$ ('*') and based on $\mathbf{T}$ ('o'), with N=25. (a) 4th-order: The error norms of $\mathbf{S}$ and $\mathbf{T}$ are 0.0719 and 0.0312, respectively. (b) 6th-order: The error norms of $\mathbf{S}$ and $\mathbf{T}$ are 0.1427 and 0.0579, respectively. (c) 8th-order: The error norms of $\mathbf{S}$ and $\mathbf{T}$ are 0.2637 and 0.0959, respectively. (d) 10th-order: The error norms of $\mathbf{S}$ and $\mathbf{T}$ are 0.4965 and 0.1472, respectively.

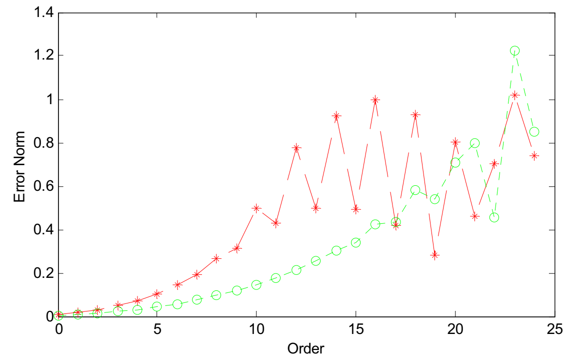

Fig. 2. The error norms of the discrete Hermite-Gaussian like functions based on $\mathbf{S}$ ('*'), and the discrete Hermite-Gaussian like functions based on $\mathbf{T}$ ('o') of various orders, with N=25.

Computer experiment 1: Fig. 1 (a)-(d) show the 4th, 6th, 8th, and 10th-orders continuous Hermite-Gaussian functions, the discrete Hermite-Gaussian like functions based on $\mathbf{S}$, and the discrete Hermite-Gaussian like functions based on $\mathbf{T}$, with N=25. The error norms, which are the Euclidean norms of the error vectors between the discrete Hermite-Gaussian like functions based on $\mathbf{S}$ (or $\mathbf{T}$) and samples of the continuous Hermite-Gaussian functions, are plotted in Fig. 2 (with N=25). Fig.1 and Fig. 2 both demonstrate that the discrete Hermite-Gaussian like functions based on $\mathbf{T}$ are better than those based on $\mathbf{S}$. The error norms of the discrete Hermite-Gaussian like functions based on both $\mathbf{S}$ and $\mathbf{T}$ tend to increase for higher order ones because of the aliasing effects.

From the definition of $\mathbf{T}$ in (8), we can express $\mathbf{T}$ in block matrix form as:

1) If N is odd,

$$\mathbf{T} = \begin{bmatrix} 1 & 0.5\mathbf{e}_1^T & 0.5\mathbf{e}_1^T\mathbf{J} \\ 0.5\mathbf{e}_1 & \mathbf{T}_1 & \mathbf{A} \\ 0.5\mathbf{J}\mathbf{e}_1 & \mathbf{JAJ} & \mathbf{JT}_1\mathbf{J} \end{bmatrix}, \tag{9}$$

with $\mathbf{e}_1$ being $[1,0,\cdots,0]^T$ of size (N-1)/2, and $\mathbf{T}_1$ and $\mathbf{A}$ being the $\frac{N-1}{2}\times\frac{N-1}{2}$ submatrices of $\mathbf{T}$. $\mathbf{J}$ is the exchange matrix with ones on the antidiagonal.

2) If N is even,

$$\mathbf{T} = \begin{bmatrix} 1 & 0.5\mathbf{e}_1^T & 0 & 0.5\mathbf{e}_1^T\mathbf{J} \\ 0.5\mathbf{e}_1 & \mathbf{T}_2 & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & 0 & \mathbf{0} \\ 0.5\mathbf{J}\mathbf{e}_1 & \mathbf{0} & \mathbf{0} & \mathbf{JT}_2\mathbf{J} \end{bmatrix}, \tag{10}$$

with $\mathbf{e}_1$ being the vector $[1,0,\cdots,0]^T$ of size $(\frac{N}{2}-1)$, and $\mathbf{T}_2$ being the $(\frac{N}{2}-1)\times(\frac{N}{2}-1)$ submatrix of $\mathbf{T}$.

Then, we have the following property.

Property 2: For the $N\times N$ $\mathbf{T}$ matrix defined in (8), the transformed matrix

$$\overline{\mathbf{T}} = \mathbf{UTU} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \tag{11}$$

is a block diagonal matrix, where $\mathbf{U}$ is the $N\times N$ unitary symmetric matrix defined by

$$\mathbf{U} = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\frac{N-1}{2}} & \mathbf{J}_{\frac{N-1}{2}} \\ \mathbf{0} & \mathbf{J}_{\frac{N-1}{2}} & -\mathbf{I}_{\frac{N-1}{2}} \end{bmatrix}, & \text{if } N \text{ is odd,} \\ \dfrac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\frac{N}{2}-1} & \mathbf{0} & \mathbf{J}_{\frac{N}{2}-1} \\ 0 & \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\frac{N}{2}-1} & \mathbf{0} & -\mathbf{I}_{\frac{N}{2}-1} \end{bmatrix}, & \text{if } N \text{ is even,} \end{cases} \tag{12}$$

with $\mathbf{J}_q$ being the $q\times q$ exchange matrix, and $\mathbf{M}_1$ and $\mathbf{M}_2$ are two square matrices of sizes $\lfloor \frac{N}{2}+1 \rfloor$ and $\lfloor \frac{N-1}{2} \rfloor$, respectively. $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Moreover, from (9) and (10), $\mathbf{M}_1$ and $\mathbf{M}_2$ in (11) are respectively

$$\mathbf{M}_1 = \begin{cases} \begin{bmatrix} 1 & \frac{1}{\sqrt{2}}\mathbf{e}_1^T \\ \frac{1}{\sqrt{2}}\mathbf{e}_1 & \mathbf{T}_1 + \mathbf{AJ} \end{bmatrix}, & \text{if } N \text{ is odd} \\ \begin{bmatrix} 1 & \frac{\sqrt{2}}{2}\mathbf{e}_1^T & 0 \\ \frac{\sqrt{2}}{2}\mathbf{e}_1 & \mathbf{T}_2 & \mathbf{0}_{\frac{N}{2}-1} \\ 0 & \mathbf{0}_{\frac{N}{2}-1}^T & 0 \end{bmatrix}, & \text{if } N \text{ is even,} \end{cases} \tag{13}$$

with $\mathbf{0}_{\frac{N}{2}-1}$ being the $(\frac{N}{2}-1)\times 1$ zero vector, and

$$\mathbf{M}_2 = \begin{cases} \mathbf{J}_{\frac{N-1}{2}}\mathbf{T}_1\mathbf{J}_{\frac{N-1}{2}} - \mathbf{J}_{\frac{N-1}{2}}\mathbf{A}, & \text{if } N \text{ is odd} \\ \mathbf{J}_{\frac{N}{2}-1}\mathbf{T}_2\mathbf{J}_{\frac{N}{2}-1}, & \text{if } N \text{ is even.} \end{cases} \tag{14}$$

From [6], we know that any symmetric and exactly tridiagonal matrix with nonzero subdiagonal entries has distinct eigenvalues. From *Property 2*, it can then be shown that $\mathbf{M}_1$ and $\mathbf{M}_2$ has distinct eigenvalues with the exception that the zero eigenvalue of $\mathbf{M}_1$ is of multiplicity two when N is even. We can also show that most of the even extensions and odd extensions of eigenvectors of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, are eigenvectors of $\mathbf{F}$. But if N is even, the even extensions of eigenvectors of $\mathbf{M}_1$ corresponding to the zero eigenvalue are not necessarily eigenvectors of $\mathbf{F}$. Thus, for N even, we need to develop a method to compute the eigenvectors of $\mathbf{F}$ in the even subspace spanned by eigenvectors of $\mathbf{T}$ of the zero eigenvalue.

Property 3: If N is even, the two orthogonal eigenvectors of $\mathbf{F}$ in the subspace spanned by even eigenvectors of $\mathbf{T}$ of eigenvalue zero are $[1,-1,1,-1,\cdots,1,-1]^T \pm \sqrt{N}\mathbf{e}_{\frac{N}{2}+1}$, where $\mathbf{e}_{\frac{N}{2}+1}$ is the $N\times 1$ column vector with zero entries except a 1 at the $(\frac{N}{2}+1)^{\text{th}}$ entry.

3. LINEAR COMBINATIONS OF MATRICES S AND T

Property 4: If k_1 and k_2 are any two constants, then $k_1\mathbf{S}+k_2\mathbf{T}$ commutes with the DFT matrix $\mathbf{F}$, where $\mathbf{S}$ and $\mathbf{T}$ are defined in (6) and (8), respectively.

From *Property 4*, we can compute the eigenvectors of DFT matrix $\mathbf{F}$ using $k_1\mathbf{S}+k_2\mathbf{T}$. Since $k_1\mathbf{S}+k_2\mathbf{T}$ and $\mathbf{S}+(k_2/k_1)\mathbf{T}$ have the same eigenvectors if k_1 is nonzero, we discuss in the following linear combinations of $\mathbf{S}$ and $\mathbf{T}$ of the form $\mathbf{S}+k\mathbf{T}$. If $k>0$, we find from computer experiments that the eigenvalues of $\mathbf{S}+k\mathbf{T}$ are distinct. Therefore, from *Property 4*, eigenvectors of $\mathbf{S}+k\mathbf{T}$ are all eigenvectors of $\mathbf{F}$ if $k>0$. Because the eigenvectors of both $\mathbf{S}$ and $\mathbf{T}$ are discrete Hermite-Gaussian like functions, we can expect that eigenvectors of $\mathbf{S}+k\mathbf{T}$ are also discrete Hermite-Gaussian like functions. We next show through computer experiments that eigenvectors of $\mathbf{S}+k\mathbf{T}$ are new versions of discrete Hermite-Gaussian like functions and, with appropriate choice of k, these new discrete Hermite-Gaussian like functions approximate samples of the continuous Hermite-Gaussian functions better than those obtained from both $\mathbf{S}$ and $\mathbf{T}$.

Computer experiment 2: To determine the optimal choice of k, we first compute the eigenvectors of $\mathbf{S}+k\mathbf{T}$, which are new versions of discrete Hermite-Gaussian like functions. All of the resulting N eigenvectors are compared with samples of the continuous Hermite-Gaussian functions of the corresponding orders and the total error norms are calculated. For $N=25$ and $N=145$, the total error norms are plotted versus various values of k (from $k=0$ to $k=50$ with spacing 1) in Fig. 3(a) and Fig. 3(b), respectively. From these results and other experiments for different values of N (up to 145), we find that the optimal k is approximately 15.

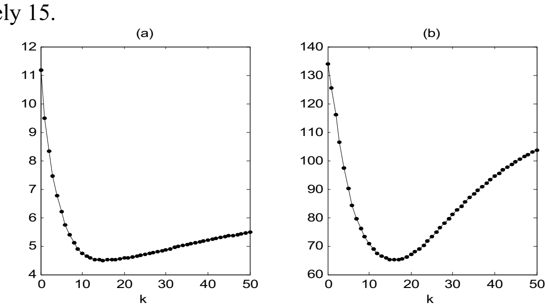

Fig. 3. Total error norms of discrete Hermite-Gaussian like functions based on $\mathbf{S}+k\mathbf{T}$. (a) $N=25$. (b) $N=145$.

4. DISCRETE FRACTIONAL FOURIER TRANSFORM BASED ON T OR S+kT AND ITS APPLICATION

The DFRFT based on $\mathbf{T}$ (or $\mathbf{S}+k\mathbf{T}$) is:

$$\mathbf{F}_{\mathbf{T}}^a = \mathbf{UD}^a\mathbf{U}^T = \begin{cases} \sum_{r=0}^{N-1} e^{-j\frac{\pi}{2}ra}\mathbf{u}_r\mathbf{u}_r^T, & \text{for } N \text{ odd} \\ \sum_{r=0}^{N-2} e^{-j\frac{\pi}{2}ra}\mathbf{u}_r\mathbf{u}_r^T + e^{-j\frac{\pi}{2}Na}\mathbf{u}_N\mathbf{u}_N^T, & \text{for } N \text{ even,} \end{cases} \tag{15}$$

where $\mathbf{U} = [\mathbf{u}_0 \mid \mathbf{u}_1 \mid \cdots \mid \mathbf{u}_{N-2} \mid \mathbf{u}_{N-1}]$ for odd N,

$\mathbf{U}=[\mathbf{u}_0\,|\,\mathbf{u}_1\,|\cdots|\,\mathbf{u}_{N-2}\,|\,\mathbf{u}_N]$ for even N, and $\mathbf{u}_r$ is the r^{th}-order discrete Hermite-Gaussian like function with r zero-crossings and is computed from the corresponding normalized eigenvector of **T** (or **S**+k**T**). The performances of the DFRFTs based on **S** and **T** (or **S**+k**T**) are compared in the following experiment.

Computer experiment 3: We compute the continuous FRT, and the DFRFTs based on **S**, **T**, and **S**+15**T** of the following rectangular function

$x(t) = 1$ when $|t| \le 17/16$, $\quad x(t) = 0$ elsewhere. (16)

The continuous FRT is computed by numerical integration of the definition of FRT in (1). The DFRFTs based on **S**, **T**, and **S**+15**T** for the samples of $x(t)$ in (16) are computed with sample number N=64 and sampling interval 1/8. The transform results of $x(t)$ are plotted in Fig. 4 with transform order a = 0.25. We find that the transform results of the DFRFTs based on **T** and **S**+15**T** are more similar to those of the continuous FRT. Their root mean square errors (RMSE) are obviously less than that of the DFRFT based on **S**.

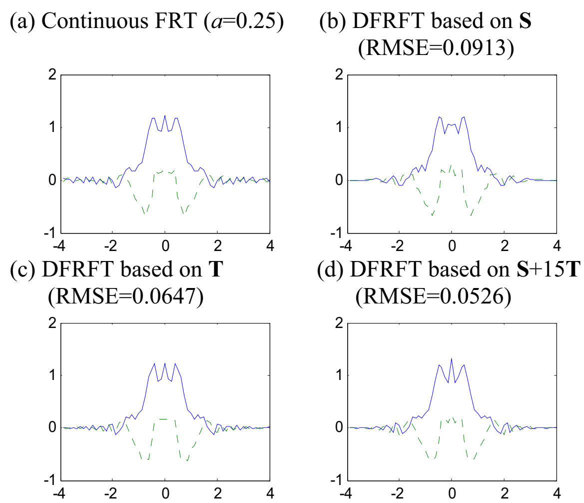

Fig. 4. Comparing the real parts (solid lines) and the imaginary parts (dashes) of the transform results of the continuous FRT and the DFRFTs based on **S**, **T**, and **S**+15**T** for a rectangular function (transform order a = 0.25).

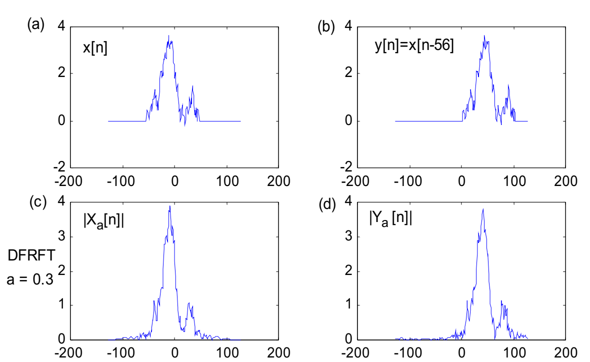

Fig. 5. If $y[n] = x[n-\tau]$, after doing the DFRFT based on **S**+15**T**, the amplitudes are the same and the distance is reduced to $\tau\cdot\cos(a\pi/2)$.

We then give an example that uses the DFRFT based on **T** or **S**+k**T** for space-variant pattern recognition. It is known that the continuous FRT has the following property [4]:

$$g(t)=f(t-\tau)$$
$$\to G_a(u)=e^{j\frac{\pi\tau^2\sin 2\alpha}{2}}e^{-j2\pi\tau u\sin\alpha}F_a(u-\tau\cos\alpha) \qquad (17)$$

where $\alpha = a\pi/2$, $F_a(u)$ and $G_a(u)$ are the continuous FRTs of $f(t)$ and $g(t)$, respectively. In other words, if $g(t)$ is the same as $f(t)$ except for the locations, then after doing the FRT, their amplitudes are also the same, and the difference of the locations is reduced by multiplying $\cos\alpha$:

$$|G_a(u)| = |F_a(u-\tau\cdot\cos\alpha)|. \qquad (18)$$

Since the DFRFT based on **T** or **S**+k**T** are very similar to the continuous FRT, the properties in (17) and (18) also apply for it with some modification:

$$g[n]=f[n-k]$$
$$\to G_a[m]\approx e^{j\frac{\pi k^2\sin 2\alpha}{2N}}e^{-j\frac{2\pi}{N}km\sin\alpha}F_a[m-R(k\cos\alpha)], \qquad (19)$$
$$|G_a[m]| \approx |F_a[m-R(k\cos\alpha)]|$$
$$\alpha = a\pi/2, \quad R(\ ): \text{rounding operation}, \qquad (20)$$

where $F_a[m]$ and $G_a[m]$ are the DFRFTs based on **T** or **S**+k**T** for $f[n]$ and $g[n]$. It can be shown from the experiments in Fig. 5. In Figs. 5(a)-(b), $x[n]$ is a signal generated by random variables and $y[n]$ is a shifting version of $x[n]$. We do the DFRFT based on **S**+15**T** of order a = 0.3 for $x[n]$ and $y[n]$ and show the amplitudes of the results in Figs. 5(c)-(d). Then we find that $|Y_a[m]|$ is very similar to the shifting of $|X_a[m]|$. From (20), the distance between $|X_a[m]|$ and $|Y_a[m]|$ should be

$$R(56\cos(0.3\pi/2)) = R(49.8964) = 50. \qquad (21)$$

From Figs. 5(c)-(d), it can be found that $|Y_a[m]|$ is indeed very near to $|X_a[m-50]|$. In fact, their correlation is near to 100%:

$$\sum_m |X_a[m]||Y_a[m+50]| \Big/ \sum_m |X_a[m]|^2 = 99.72\%. \qquad (22)$$

Thus we can use the DFRFT based on **T** or **S**+k**T** to do space-variant pattern recognition. In the transform domain, we can use whether there exists an h such that $|G_a[m]| = |F_a[m-h]|$ to conclude whether the two patterns $g[n]$ and $f[n]$ are equivalent. If so, we can use $k \approx h/\cos\alpha$ to estimate the distance between the two patterns in the space domain.

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