STEADY-STATE AND TRANSIENT ANALYSIS OF MULTICHANNEL FILTERED-X AFFINE PROJECTION ALGORITHMS

Alberto Carini

STI - University of Urbino
Piazza della Repubblica, 13 - 61029 Urbino, Italy

Giovanni L. Sicuranza

DEEI - University of Trieste
Via A. Valerio, 10 - 34127 Trieste, Italy

ABSTRACT

The paper provides an analysis of steady-state and transient behavior of two filtered-x affine projection algorithms suitable for multichannel active noise control. The analysis relies on energy conservation arguments, it does not apply the independence theory nor it imposes any restriction to the signal distributions. The paper also shows that filtered-x affine projection algorithms always provide a biased estimate of the minimum-mean-square solution of active noise control problem.

1. INTRODUCTION

The affine projection (AP) algorithms are a family of adaptive filters that can produce a good tradeoff between convergence speed and computational complexity. Their properties have been exploited for many years in the field of acoustic echo cancellation and recently it was recognized that the filtered-x AP algorithms can be very helpful also in the field of active noise control (ANC) both for single-channel and multi-channel solutions [1]. Despite the great interest for these algorithms, very few publications deal with the convergence properties of AP and filtered-x algorithms. Early convergence analysis results for AP and filtered-x algorithms were mostly based on the independence theory (IT) and constrained the probability distribution of the input signal to be Gaussian or spherically invariant [2]. The IT assumes the statistical independence of time-lagged input data vectors. While the IT works well for LMS and NLMS algorithm analysis, this hypothesis is too strong for filtered-x [3] and AP algorithms [4, 5]. Different approaches have been studied in order to overcome the IT. [3] presents an analysis of the mean weight behavior of the filtered-x LMS algorithm based only on the hypothesis of neglecting the correlation between coefficient and signal vectors. Moreover, the analysis of [3] does not impose any restriction on the signal distributions. Another analysis approach that avoids IT is applied in [4] for the mean-square performance analysis of AP algorithms. In [4] the convergence treatment relies on energy conservation arguments and no restriction is imposed on the signal distributions. Also part of the analysis of [4] is based on the hypothesis of neglecting the correlation between coefficient vectors and some signal functions. Furthermore, simple expressions are derived for the mean-square-error (MSE) and the mean-square-deviation (MSD) of the family of AP algorithms and conditions on the step-size for the mean-square stability are obtained.

In this paper we apply and adapt the approach of [4] to the steady-state and transient analysis of multichannel filtered-x AP algorithms suitable for active noise control. In particular we consider and compare the exact filtered-x AP algorithm [1] and a novel approximate algorithm. We also show in the paper that filtered-x AP algorithms always provide a biased estimate of the minimum-mean-square solution of the active noise control problem. Nevertheless, in many cases the bias is small and therefore the filtered-x AP algorithms can be profitably applied to active noise control.

The paper is organized as follows. Section 2 reviews the multichannel feedforward active noise controller structure and the multichannel filtered-x AP algorithm. Section 3 compares the minimum-mean-square solution of the ANC problem with the asymptotic solution of filtered-x AP algorithms. Section 4 presents the analysis of the steady-state and transient behavior of filtered-x AP algorithms. Section 5 provides some comparisons between theoretically predicted values and simulation results. Conclusion follows in Section 6.

Throughout the paper small boldface letters are used to denote vectors and bold capital letters are used to denote matrices, e.g. $\mathbf{x}$ and $\mathbf{X}$, all vectors are column vectors, the boldface symbol $\mathbf{I}$ indicates an identity matrix of appropriate dimensions, the symbol $\odot$ denotes the linear convolution, diag$\{\ldots\}$ is a block-diagonal matrix of the entries $\{\ldots\}$, $E[\,\cdot\,]$ denotes the mathematical expectation, $\|\cdot\|_{\mathbf{\Sigma}}$ is the weighted Euclidean norm, e.g. $\|\mathbf{w}\|_{\mathbf{\Sigma}} = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w}$ with $\mathbf{\Sigma}$ a symmetric positive definite matrix, vec$\{\cdot\}$ indicates the vector operator and vec$^{-1}\{\cdot\}$ the inverse vector operator that returns a square matrix from an input vector of appropriate dimensions, $\otimes$ denotes the Kronecker product.

This work was supported in part by the MIUR under Grant PRIN 2004092314.

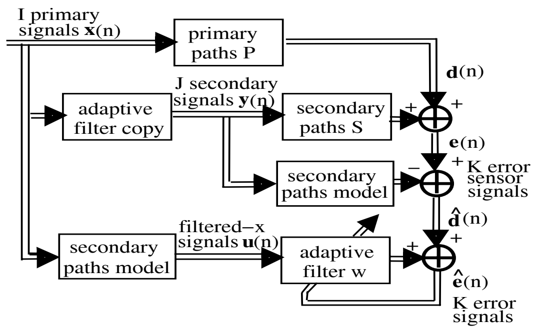

Fig. 1. Delay-compensated filtered-x structure for active noise control.

2. MULTICHANNEL FILTERED-X AFFINE PROJECTION ALGORITHMS

Multichannel active noise controllers are based on the destructive interference in given locations of the noise produced by some primary sources and the interfering signal generated by some secondary sources. Fig. 1 shows the block diagram of a multichannel delay-compensated filtered-x active noise control system. As usual, the primary and secondary paths, which propagate the primary and secondary source signals, respectively, are modelled with linear FIR filters. In order to compensate for the propagation delay introduced by the secondary paths, the output of the primary paths $\mathbf{d}(n)$ is estimated by subtracting the output of the secondary paths model from the error sensors signals $\mathbf{e}(n)$. In this paper we assume perfect modelling of the secondary paths, i.e. $\hat{\mathbf{d}}(n) = \mathbf{d}(n)$. In the adaptive filter any input i is connected to any output j with a linear filter.

The following notation is used throughout the paper:
I is the number of primary source signals,
J is the number of secondary source signals,
K is the number of error sensors,
L is the affine projection order,
$s_{k,j}(n)$ is the impulse response of the secondary path that connects the j-th secondary source to the k-th error sensor,
$\mathbf{w}_{j,i}(n)$ is the coefficient vector of the FIR filter that connects the input i to the output j of the adaptive controller,
$\mathbf{x}_i(n)$ is the i-th primary source input signal vector,
$\mathbf{x}(n) = \big[\mathbf{x}_1^T(n), \dots, \mathbf{x}_I^T(n)\big]^T$,
$\mathbf{w}_j(n) = \big[\mathbf{w}_{j,1}^T(n), \dots, \mathbf{w}_{j,I}^T(n)\big]^T$,
$y_j(n) = \mathbf{w}_j^T(n)\mathbf{x}(n)$ is the j-th secondary source signal,
$d_k(n)$ is the output of the k-th primary path,
$\mathbf{w}(n) = \big[\mathbf{w}_1^T(n), \dots, \mathbf{w}_J^T(n)\big]^T$,
M is the total number of coefficients of $\mathbf{w}(n)$,
$\mathbf{u}_k(n) = \big[s_{k,1}(n) \odot \mathbf{x}^T(n), \dots, s_{k,J}(n) \odot \mathbf{x}^T(n)\big]^T$,
$\mathbf{d}_k(n) = \big[d_k(n), \dots, d_k(n-L+1)\big]^T$,
$\mathbf{d}(n) = \big[\mathbf{d}_1^T(n), \dots, \mathbf{d}_K^T(n)\big]^T$,
$\mathbf{U}_k(n) = \big[\mathbf{u}_k(n), \dots, \mathbf{u}_k(n-L+1)\big]$,
$\mathbf{U}(n) = \big[\mathbf{U}_1(n), \dots, \mathbf{U}_K(n)\big]$,
$\mathbf{e}(n) = \mathbf{d}(n) + \mathbf{U}^T(n)\mathbf{w}(n)$.

With this notation the adaptation rule of the filtered-x AP algorithm [1] can be written as in equation (1),

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \mu\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{e}(n), \tag{1}$$

where the $KL \times KL$ matrix $\mathbf{V}(n)$ is given in equation (2),

$$\mathbf{V}(n) = \mathbf{U}^T(n)\mathbf{U}(n) + \delta\mathbf{I}, \tag{2}$$

with δ a small positive regularization term. We consider also a novel approximate algorithm obtained by estimating $\mathbf{V}(n)$ with the expression of equation (3),

$$\mathbf{V}(n) = \mathrm{diag}\Big\{\mathbf{U}_1^T(n)\mathbf{U}_1(n)+\delta\mathbf{I}, \dots, \mathbf{U}_K^T(n)\mathbf{U}_K(n)+\delta\mathbf{I}\Big\}. \tag{3}$$

This equation provides a less computationally intensive updating rule since the interactions among the error signals are not considered.

By substituting the expression of $\mathbf{e}(n)$ in equation (1), we obtain the expression of equation (4) that will be used for the algorithms analysis,

$$\mathbf{w}(n+1) = \big(\mathbf{I} - \mu\mathbf{P}(n)\big)\mathbf{w}(n) - \mu\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{d}(n), \tag{4}$$

with $\mathbf{P}(n) = \mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{U}^T(n)$.

3. MINIMUM-MEAN-SQUARE SOLUTION AND ASYMPTOTIC SOLUTION

The minimum-mean-square solution, $\mathbf{w}_o$, of the active noise control problem is given by equation (5),

$$\mathbf{w}_o = -\mathbf{R}_{uu}^{-1}\mathbf{R}_{ud}, \tag{5}$$

where $\mathbf{R}_{uu} = E\big[\sum_{k=1}^{K}\mathbf{u}_k(n)\mathbf{u}_k^T(n)\big]$ and $\mathbf{R}_{ud} = E\big[\sum_{k=1}^{K}\mathbf{u}_k(n)d_k(n)\big]$. On the contrary, from equation (4) it can be easily deduced that the multichannel AP algorithms, when converging, tend asymptotically to the coefficient vector of equation (6),

$$\mathbf{w}_\infty = -E\big[\mathbf{P}(n)\big]^{-1}E\big[\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{d}(n)\big]. \tag{6}$$

The expression in (6) can also be written as

$$\mathbf{w}_\infty = \mathbf{w}_o - E\big[\mathbf{P}(n)\big]^{-1}E\big[\mathbf{U}(n)\mathbf{V}^{-1}(n)\boldsymbol{\nu}_o(n)\big], \tag{7}$$

where $\boldsymbol{\nu}_o(n) = \mathbf{d}(n) + \mathbf{U}^T(n)\mathbf{w}_o$ is the optimal residual error. The orthogonality principle here imposes that $E\big[\sum_{k=1}^{K}\mathbf{u}_k(n)\big(d_k(n) + \mathbf{u}_k^T(n)\mathbf{w}_o\big)\big] = 0$ but this condition is not sufficient to guarantee $E\big[\mathbf{U}(n)\mathbf{V}^{-1}(n)\boldsymbol{\nu}_o(n)\big]$ to be zero. Indeed, in active noise control systems, the secondary paths are often non-minimum-phase and do not admit a causal inverse system [6]. In such cases, the active noise controller operates as a predictor [6]. Moreover, most multichannel active noise control problems, even in absence of measurement noise, do not have an exact solution. For these reasons the optimal residual error $\boldsymbol{\nu}_o(n)$ is often colored and correlated with $\mathbf{U}(n)$. Consequently the asymptotic solution of the AP algorithms in (6) differs from the minimum-mean-square solution in (5).

4. STEADY-STATE AND TRANSIENT ANALYSIS

For the transient analysis we are interested in the time evolution of $E[\|\tilde{\mathbf{w}}(n)\|_{\boldsymbol{\Sigma}}]$ for a suitable coefficient error vector $\tilde{\mathbf{w}}(n)$ and for some appropriate choices of the symmetric positive definite matrix $\boldsymbol{\Sigma}$. Usually the coefficient error vector $\tilde{\mathbf{w}}(n)$ is defined as the difference between $\mathbf{w}(n)$ and the minimum-mean-square solution, i.e. $\tilde{\mathbf{w}}(n)$=$\mathbf{w}(n)$-$\mathbf{w}_o$. Indeed, in most of the approaches presented in literature the choice of working with $\mathbf{w}_o$ is motivated by the fact that, thanks to the orthogonality principle, a simple updating relation for $E\left[\|\tilde{\mathbf{w}}(n)\|^2_{\boldsymbol{\Sigma}}\right]$ can be easily derived. In our case the filtered-x AP algorithms do not converge to $\mathbf{w}_o$ and the residual error $\boldsymbol{\nu}_o(n)$ is colored and correlated with $\mathbf{U}(n)$. For these reasons, we do not have here any advantage from the use of $\mathbf{w}_o$ in $\tilde{\mathbf{w}}(n)$ and we prefer to define $\tilde{\mathbf{w}}(n)$=$\mathbf{w}(n)$-$\mathbf{w}_a$, where the auxiliary vector $\mathbf{w}_a$ is given by equation (8),

$$\mathbf{w}_a = -E\big[\big(\mathbf{I}-\mu\mathbf{P}(n)\big)\,\boldsymbol{\Sigma}\,\mathbf{P}(n)\big]^{-1}\cdot E\big[\big(\mathbf{I}-\mu\mathbf{P}(n)\big)\,\boldsymbol{\Sigma}\,\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{d}(n)\big]. \quad (8)$$

With this choice of the auxiliary vector $\mathbf{w}_a$ an efficient recursion can be derived for $E[\|\tilde{\mathbf{w}}(n)\|_{\boldsymbol{\Sigma}}]$. By assuming $\tilde{\mathbf{w}}(n)$ to be uncorrelated with $\mathbf{P}(n)$ and with $\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{d}(n)$, it can be shown from equations (4) and (8) that

$$E[\|\tilde{\mathbf{w}}(n+1)\|_{\boldsymbol{\Sigma}}] = E[\|\tilde{\mathbf{w}}(n)\|_{\boldsymbol{\Sigma}'}]+ \mu^2 E[\boldsymbol{\nu}_a^T(n)\mathbf{V}^{-1}(n)\mathbf{U}^T(n)\,\boldsymbol{\Sigma}\,\mathbf{U}(n)\mathbf{V}^{-1}(n)\boldsymbol{\nu}_a(n)], \quad (9)$$

where $\boldsymbol{\Sigma}' = E[\big(\mathbf{I}-\mu\mathbf{P}(n)\big)\,\boldsymbol{\Sigma}\,\big(\mathbf{I}-\mu\mathbf{P}(n)\big)\,]$ and $\boldsymbol{\nu}_a(n) = \mathbf{d}(n) + \mathbf{U}^T(n)\mathbf{w}_a$. By using the inverse vector operator, equation (9) can be written in the form of equation (10),

$$E[\|\tilde{\mathbf{w}}(n{+}1)\|_{\text{vec}^{-1}\{\boldsymbol{\sigma}\}}] = E[\|\tilde{\mathbf{w}}(n)\|_{\text{vec}^{-1}\{\mathbf{F}\boldsymbol{\sigma}\}}]{+}\mu^2\boldsymbol{\gamma}_a^T\boldsymbol{\sigma}, \quad (10)$$

where $\boldsymbol{\sigma} = \text{vec}\{\boldsymbol{\Sigma}\}$, $\mathbf{F} = \mathbf{I}-\mu(E[\mathbf{P}(n)]\otimes\mathbf{I}+\mathbf{I}\otimes E[\mathbf{P}]) + \mu^2 E[\mathbf{P}(n)\otimes\mathbf{P}(n)]$ and $\boldsymbol{\gamma}_a = \text{vec}\{E[\mathbf{U}(n)\mathbf{V}^{-1}(n)\boldsymbol{\nu}_a(n)\cdot \boldsymbol{\nu}_a^T(n)\mathbf{V}^{-1}(n)\mathbf{U}^T(n)]\}$. For compactness, in what follows we will drop the notation $\text{vec}^{-1}\{\cdot\}$ from the subscript of the weighted Euclidean norm and we will keep only the vectors $\boldsymbol{\sigma}$ and $\mathbf{F}\boldsymbol{\sigma}$.

By following the same derivations of [4] we can arrive to the following result:

Under the assumption that $\tilde{\mathbf{w}}(n)$ is uncorrelated with $\mathbf{P}(n)$ and with $\mathbf{U}(n)\mathbf{V}^{-1}(n)\mathbf{d}(n)$, the transient behavior of the filtered-x AP algorithms with updating rule given by equations (1) is described by the state recursion

$$\mathcal{W}(n+1) = \mathcal{F}\,\mathcal{W}(n) + \mu^2\mathcal{Y} \quad (11)$$

where

$$\mathcal{F} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{M^2-1} \end{bmatrix},$$

$$\mathcal{W}(n) = \begin{bmatrix} E[\|\tilde{\mathbf{w}}(n)\|_{\boldsymbol{\sigma}} \\ E[\|\tilde{\mathbf{w}}(n)\|_{\mathbf{F}\boldsymbol{\sigma}} \\ \vdots \\ E[\|\tilde{\mathbf{w}}(n)\|_{\mathbf{F}^{M^2-1}\boldsymbol{\sigma}} \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} \gamma_a^T\boldsymbol{\sigma} \\ \gamma_a^T\mathbf{F}\boldsymbol{\sigma} \\ \vdots \\ \gamma_a^T\mathbf{F}^{M^2-1}\boldsymbol{\sigma} \end{bmatrix},$$

and the p_i are the coefficients of the characteristic polynomial of $\mathbf{F}$, i.e. $p(x) = x^{M^2} + p_{M^2-1}x^{M^2-1} + \ldots + p_1 x + p_0 = \det(x\mathbf{I}-\mathbf{F})$.

A part from the different definition of the involved matrices, the result of equation (11) is the same of Theorem 1 of [4]. Therefore the same conclusions of [4] can be applied for the mean-square stability of the algorithm.

We are also interested in the values of the mean-square-error (MSE) and the mean-square-deviation (MSD) at steady-state. It can be proved that these values can be estimated with the expressions of equations (12) and (13), respectively,

$$\lim_{n\to\infty} E[\textstyle\sum_{k=1}^K \|\mathbf{d}(n)+\mathbf{w}^T(n)\mathbf{u}_k(n)\|^2] = \lim_{n\to\infty} E[\|\mathbf{w}(n)-\mathbf{w}_a\|^2_{\mathbf{R}_{uu}}] + E[\textstyle\sum_{k=1}^K \|d_k(n)\|^2] + 2\mathbf{w}_\infty^T\mathbf{R}_{ud} + 2\mathbf{w}_a^T\mathbf{R}_{uu}\mathbf{w}_\infty - \mathbf{w}_a^T\mathbf{R}_{uu}\mathbf{w}_a, \quad (12)$$

$$\lim_{n\to\infty} E[\|\mathbf{w}(n)-\mathbf{w}_\infty\|^2] = \lim_{n\to\infty} E[\|\mathbf{w}(n)-\mathbf{w}_a\|^2_{\mathbf{I}}\,] + \|\mathbf{w}_a-\mathbf{w}_\infty\|^2, \quad (13)$$

where, similarly to [4],

$$\lim_{n\to\infty} E[\|\mathbf{w}(n)-\mathbf{w}_a\|^2_{\boldsymbol{\Sigma}}\,] = \mu^2\boldsymbol{\gamma}_a(\mathbf{I}-\mathbf{F})^{-1}\boldsymbol{\sigma}, \quad (14)$$

with $\boldsymbol{\Sigma} = \mathbf{R}_{uu}$ in equation (12) and $\boldsymbol{\Sigma} = \mathbf{I}$ in equation (13).

5. EXPERIMENTAL RESULTS

In this section we show some experimental results obtained with a multichannel active noise control system with $I = 1$, $J = 2$, $K = 2$. The impulse responses of the primary paths were $\mathbf{p}_{11}(n) = [0, 0, 1, -0.3, 0.2]$ and $\mathbf{p}_{21}(n) = [0, 0, 1, -0.2, 0.1]$, while those of the secondary paths were $\mathbf{s}_{11}(n) = [0, 2, -0.5, 0.1]$, $\mathbf{s}_{12}(n) = [0, 2, -0.3, -0.1]$,

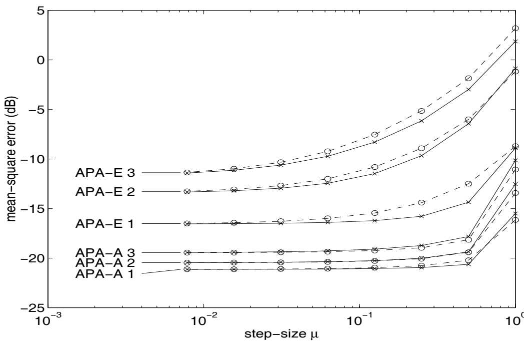

Fig. 2. Theoretical (–) and simulation values (- -) of steady-state MSE of the exact (APA-E) and approximate (APA-A) algorithms for AP orders $L = 1, 2$ and 3 versus step-size.

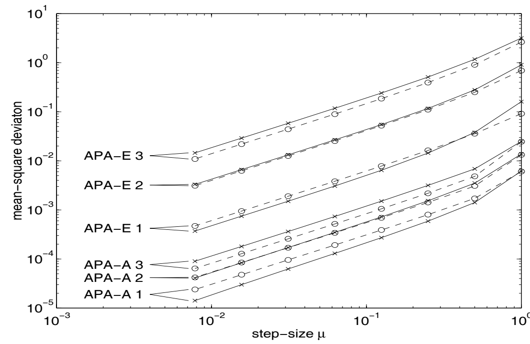

Fig. 3. Theoretical (–) and simulation values (- -) of steady-state MSD of the exact (APA-E) and approximate (APA-A) algorithms for AP orders $L = 1, 2$ and 3 versus step-size.

Table 1. Minimum-mean-square and asymptotic solutions of the exact (APA-E) and approximate (APA-A) algorithms for AP orders $L = 1, 2$ and 3.

$\mathbf{w}_o$	$\mathbf{w}_\infty$ APA-E			$\mathbf{w}_\infty$ APA-A		
	L=1	L=2	L=3	L=1	L=2	L=3
0.93	0.97	0.91	0.88	0.92	0.85	0.84
-0.71	-0.79	-0.67	-0.66	-0.72	-0.67	-0.66
0.21	0.31	0.23	0.27	0.24	0.19	0.15
-0.06	-0.13	-0.12	-0.14	-0.08	-0.05	-0.04
-0.93	-0.88	-0.71	-0.66	-0.92	-0.85	-0.83
0.27	0.21	0.06	0.12	0.25	0.17	0.16
-0.25	-0.27	-0.26	-0.28	-0.23	-0.18	-0.15
-0.03	0.01	0.02	0.00	-0.03	-0.05	-0.07

$\mathbf{s}_{21}(n) = [0, 1, -0.7, -0.2], \mathbf{s}_{22}(n) = [0, 1, -0.2, 0.2]$. The input signal was a zero mean, unit variance colored Gaussian noise with $E[x(n)x(n-m)] = 0.9^{|m|}$ and a zero mean, white Gaussian noise was added to $d_k(n)$ to get a 40 dB signal-to-noise ratio. The controller was a two-channel linear filter with memory length 4, i.e. with M=8. Table 1 provides the minimum-mean-square and the asymptotic solutions of the exact and approximate filtered-x AP algorithms with two-digits precision. The norm of the error between $\mathbf{w}_o$ and $\mathbf{w}_\infty$ in this case increases with the AP order L. In fact, due to the correlation between $\boldsymbol{\nu}_o(n)$ and $\mathbf{U}(n)$, in equation (7) the norm of $E\left[\mathbf{U}(n)\mathbf{V}^{-1}(n)\boldsymbol{\nu}_o(n)\right]$ increases with the AP order. Fig. 2 and Fig. 3 diagram the MSE and the MSD of the algorithms, estimated with equations (12) and (13) or obtained from simulations, at different values of step-size μ and for the AP order $L = 1, 2$ and 3. In Fig. 2 and Fig. 3 the theoretical values of MSE and MSD fall close to the corresponding simulation values. Depending on the AP order L and on the step-size μ, the estimation errors can assume both positive or negative values. The approximate algorithm provides a lower MSE and MSD than the exact algorithm but we must point out that it provides also a lower convergence speed.

6. CONCLUSION

In the paper we have provided an analysis of steady-state and transient behavior of a couple of multichannel filtered-x AP algorithms. The analysis relies on energy conservation arguments and it does not apply IT nor it imposes any restriction to the signal distributions. We have also shown that filtered-x AP algorithms always provide a biased estimate of the minimum-mean-square solution of active noise control problem. In many cases the bias is small and therefore the filtered-x AP algorithms can be profitably applied to active noise control.

7. REFERENCES

[1] M. Bouchard, "Multichannel affine and fast affine projection algorithms for active noise control and acoustic equalization systems," *IEEE Trans. Speech and Audio Processing*, vol. 11, no. 1, pp. 54–60, Jan. 2003.

[2] E. Bjarnason, "Analysis of the filtered-X LMS algorithm," *IEEE Trans. Speech and Audio Processing*, vol. 3, no. 6, pp. 504–514, Nov. 1995.

[3] O. J. Tobias, J. M. Bermudez, and N. J. Bershad, "Mean weight behavior of the filtered-X LMS algorithm," *IEEE Trans. Signal Processing*, vol. 48, no. 4, pp. 1061–1075, Apr. 2000.

[4] H. C. Shin and A. H. Sayed, "Mean-square performance of a family of affine projection algorithms," *IEEE Trans. Signal Processing*, vol. 52, no. 1, pp. 90–102, Jan. 2004.

[5] A. H. Sayed, *Fundamentals of Adaptive Filtering*, Wiley, New York, 2003.

[6] P. Strauch and B. Mulgrew, "Active control of nonlinear noise processes in a linear duct," *IEEE Trans. Signal Processing*, vol. 46, no. 9, pp. 2404–2412, Sept. 1998.