SEQUENTIAL ESTIMATION OF STOCHASTIC CONTINUOUS-TIME SIGNALS FROM SAMPLE COVARIANCES

Magnus Mossberg

Karlstad University
Department of Electrical Engineering
SE-651 88 Karlstad, Sweden

ABSTRACT

A method for sequential estimation of stochastic continuous-time signal parameters is presented. The parameters are estimated by fitting the covariance function of the continuous-time process to sample covariances from discrete-time data in a sequential algorithm. Compact expressions are given for the sequential estimation algorithm, in which the continuous-time parameterization is kept.

1. INTRODUCTION

When describing stochastic signals and systems in a continuous-time framework, it is important to have fast and reliable estimators for the involved parameters, [1–5]. Continuous-time stochastic signal and system descriptions are often used in econometrics, [6], and in science, where it is important that the parameters have physical interpretations, see, e.g., [7]. The continuous-time stochastic descriptions are also frequently used when filter and control designs are made in continuous-time, [8], and when data are sampled irregularly, [9]. A very general continuous-time signal description is the continuous-time ARMA process, defined as

$$A(p)y(t) = B(p)e(t) \tag{1}$$

$$\mathrm{E}\{e(t)e(s)\} = \sigma_e^2\delta(t-s), \tag{2}$$

where

$$A(p) = p^n + a_1p^{n-1} + \ldots + a_n, \tag{3}$$

$$B(p) = b_0p^m + b_1p^{m-1} + \ldots + b_m, \tag{4}$$

where p denotes the differentiation operator, and $n > m$, and $A(p)$ has all zeros in the left half-plane. The continuous-time ARMA process can be interpreted as the underlying process for the spectrum

$$\phi(\mathrm{i}\omega) = \sigma_e^2\frac{|B(\mathrm{i}\omega)|^2}{|A(\mathrm{i}\omega)|^2}. \tag{5}$$

By varying the parameters $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=0}^m$, the continuous-time ARMA process has the ability to describe signals with spectra of almost any shape. Note that a continuous-time AR process is obtained as a special case if $b_i = 0$, $i = 0, \ldots, m-1$. When introducing a state space representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}e(t), \tag{6}$$

$$y(t) = \mathbf{C}^T\mathbf{x}(t) \tag{7}$$

for the continuous-time ARMA process, one possible choice of $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$ and $\mathbf{C} \in \mathbb{R}^{n\times 1}$ is

$$\mathbf{A} = \begin{bmatrix} -a_1 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -a_n & & & \end{bmatrix}, \tag{8}$$

$$\mathbf{B} = \big[0, \ldots, 0, b_0, \ldots, b_m\big]^T, \tag{9}$$

$$\mathbf{C} = \big[1, 0, \ldots, 0\big]^T, \tag{10}$$

which gives an observable canonical form. By using a Wiener process

$$w(t) = \int_0^t e(s)\,\mathrm{d}s, \tag{11}$$

the continuous-time stochastic state space description (6) can be expressed as the stochastic differential equation [10]

$$\mathrm{d}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)\,\mathrm{d}t + \mathbf{B}\,\mathrm{d}w(t), \tag{12}$$

where $\mathrm{d}w(t) = e(t)\,\mathrm{d}t$ is the increment of the Wiener process $w(t)$.

The output signal $y(t)$ is observed at $t = h, 2h, \ldots$ and the problem of interest in the paper is to estimate the continuous-time ARMA parameters

$$\boldsymbol{\theta} = \big[a_1, \ldots, a_n, b_1, \ldots, b_m\big]^T \tag{13}$$

sequentially from the discrete-time data. A possible solution is to sample the continuous-time ARMA process to get a discrete-time description. The parameters in the discrete-time description could then be estimated and mapped onto the continuous-time parameters. However, there is no closed-form expression for the mapping of the zeros, [11], which is a serious drawback for this approach, especially if it was to be implemented sequentially for a real-time scenario. This problem is circumvented if the continuous-time parameterization is kept. One way of doing this is to replace the differentiation operator in the continuous-time description with an approximation, form a linear regression and estimate the parameters using the least squares method. This is done for continuous-time AR processes in [12, 13]. Unfortunately, this approach is not straightforward to apply for a continuous-time ARMA process. The solution presented in this paper is to use an expression for the covariance function of the continuous-time ARMA process, parameterized by the continuous-time parameters

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$\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=0}^m$. The parameters are then determined by fitting the parameterized covariance function to a covariance function estimated from discrete-time data in a sequential estimation algorithm.

2. COVARIANCE FUNCTION

The covariance matrix $\mathbf{R_x}$ of the state vector $\mathbf{x}(t)$ is given from a continuous-time Lyapunov equation [14]

$$\mathbf{A}\mathbf{R_x} + \mathbf{R_x}\mathbf{A}^T + \sigma_e^2\mathbf{B}\mathbf{B}^T = \mathbf{0}. \tag{14}$$

The covariance function $\boldsymbol{\mathcal{R}}_\mathbf{x}(\tau)$ of $\mathbf{x}(t)$ is evaluated, using the solution to (6), as

$$\begin{aligned}\boldsymbol{\mathcal{R}}_\mathbf{x}(\tau) &= \mathrm{E}\{\mathbf{x}(t+\tau)\mathbf{x}^T(t)\} \\ &= \mathrm{E}\{(e^{\mathbf{A}\tau}\mathbf{x}(t) + \int_t^{t+\tau} e^{\mathbf{A}(t+\tau-s)}\mathbf{B}e(s)\mathrm{d}s)\mathbf{x}^T(t)\} \\ &= e^{\mathbf{A}\tau}\mathbf{R_x}.\end{aligned} \tag{15}$$

This gives, together with (7), that the covariance function of $y(t)$ can be expressed as

$$r_{\boldsymbol{\theta}}(\tau) = \mathrm{E}\{y(t+\tau)y^T(t)\} = \mathbf{C}^T\boldsymbol{\mathcal{R}}_\mathbf{x}(\tau)\mathbf{C}. \tag{16}$$

Here, the dependency on the parameter vector $\boldsymbol{\theta}$ in (13) is emphasized. The theoretical expression (16) for the covariance function is used in the next section, together with sample covariances, for estimating the parameter vector $\boldsymbol{\theta}$ sequentially.

3. SEQUENTIAL ESTIMATION

An estimate $\hat{r}_N(\tau)$ of the covariance function from the N samples $y(1), \ldots, y(N)$ is given as

$$\hat{r}_N(\tau) = \frac{1}{N-\tau}\sum_{t=1}^{N-\tau} y(t)y(t+\tau), \quad \tau \geq 0. \tag{17}$$

When a new sample $y(N+1)$ is added, the estimate (17) is modified sequentially as

$$\hat{r}_{N+1}(\tau) = \alpha(\tau)\hat{r}_N(\tau) + \beta(\tau)\rho(\tau) \tag{18}$$

to give an updated estimate $\hat{r}_{N+1}(\tau)$, where

$$\alpha(\tau) = \frac{N-\tau}{N+1-\tau}, \tag{19}$$

$$\beta(\tau) = \frac{1}{N+1-\tau}, \tag{20}$$

$$\rho(\tau) = y(N-\tau+1)y(N+1). \tag{21}$$

Define the loss function

$$J_N(\boldsymbol{\theta}) = \sum_{\tau=0}^{\tau_{\max}} (\hat{r}_N(\tau) - r_{\boldsymbol{\theta}}(\tau))^2, \tag{22}$$

where $\hat{r}_N(\tau)$ is defined in (17) and $r_{\boldsymbol{\theta}}(\tau)$ in (16), from which an estimate $\hat{\boldsymbol{\theta}}_N$ is obtained as

$$\hat{\boldsymbol{\theta}}_N = \arg\min_{\boldsymbol{\theta}} J_N(\boldsymbol{\theta}). \tag{23}$$

Next, it is shown how the estimator (23) can be implemented in a sequential form, where the current estimate is updated and modified when a new data point $y(N+1)$ is added. In the remainder of this section, the function argument τ is omitted.

Using (18), the loss function $J_{N+1}(\boldsymbol{\theta})$ can be expressed as

$$\begin{aligned}J_{N+1}(\boldsymbol{\theta}) &= \sum_{\tau=0}^{\tau_{\max}} (\hat{r}_{N+1} - r_{\boldsymbol{\theta}})^2 \\ &= \sum_{\tau=0}^{\tau_{\max}} \{\alpha^2\hat{r}_N^2 + \beta^2\rho^2 + r_{\boldsymbol{\theta}}^2 + 2\alpha\beta\rho\hat{r}_N \\ &\quad - 2\alpha\hat{r}_N r_{\boldsymbol{\theta}} - 2\beta\rho r_{\boldsymbol{\theta}}\} \\ &= \frac{1}{2}J_N(\boldsymbol{\theta}) + \sum_{\tau=0}^{\tau_{\max}}\{\frac{1}{2}(r_{\boldsymbol{\theta}} - 2\beta\rho)^2 - \beta^2\rho^2 \\ &\quad + (\alpha^2 - \frac{1}{2})\hat{r}_N^2 + (1-2\alpha)\hat{r}_N r_{\boldsymbol{\theta}} + 2\alpha\beta\rho\hat{r}_N\},\end{aligned} \tag{24}$$

where a completion of squares is carried out in the last step. This is not needed for the remainder of the derivation, but it gives an expression that more clearly reveals that $\boldsymbol{\theta}$ is affected by the new data point $y(N+1)$ in ρ. However, the most important fact about (24) is that $J_{N+1}(\boldsymbol{\theta})$ is explicitly dependent on $J_N(\boldsymbol{\theta})$.

Approximating $r_{\boldsymbol{\theta}}$ using a first order Taylor series expansion about $\hat{\boldsymbol{\theta}}_N$ gives

$$r_{\boldsymbol{\theta}} \approx r_{\hat{\boldsymbol{\theta}}_N} + \boldsymbol{\psi}_N^T \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N), \tag{25}$$

where

$$\boldsymbol{\psi}_N = \left.\frac{\mathrm{d}r_{\boldsymbol{\theta}}}{\mathrm{d}\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_N}. \tag{26}$$

Substituting (25) in (24) gives

$$\begin{aligned}J_{N+1}(\boldsymbol{\theta}) &\approx \frac{1}{2}J_N(\boldsymbol{\theta}) \\ &\quad + \sum_{\tau=0}^{\tau_{\max}}\{\frac{1}{2}(r_{\hat{\boldsymbol{\theta}}_N} + \boldsymbol{\psi}_N^T \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N) - 2\beta\rho)^2 \\ &\quad + (1-2\alpha)\hat{r}_N(r_{\hat{\boldsymbol{\theta}}_N} + \boldsymbol{\psi}_N^T \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N)) \\ &\quad - \beta^2\rho^2 + (\alpha^2 - \frac{1}{2})\hat{r}_N^2 + 2\alpha\beta\rho\hat{r}_N\}.\end{aligned} \tag{27}$$

Differentiation of (27) with respect to $\boldsymbol{\theta}$ gives

$$\begin{aligned}\dot{\mathbf{J}}_{N+1}(\boldsymbol{\theta}) &\approx \frac{1}{2}\dot{\mathbf{J}}_N(\boldsymbol{\theta}) \\ &\quad + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N(r_{\hat{\boldsymbol{\theta}}_N} + \boldsymbol{\psi}_N^T \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N) - 2\beta\rho) \\ &\quad + (1-2\alpha)\hat{r}_N\boldsymbol{\psi}_N\},\end{aligned} \tag{28}$$

where

$$\dot{\mathbf{J}}_N(\boldsymbol{\theta}) = \frac{\mathrm{d}J_N(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}}. \tag{29}$$

A first order Taylor series expansion of $\dot{\mathbf{J}}_N(\boldsymbol{\theta})$ around $\hat{\boldsymbol{\theta}}_N$ gives

$$\begin{aligned}\dot{\mathbf{J}}_N(\boldsymbol{\theta}) &\approx \dot{\mathbf{J}}_N(\hat{\boldsymbol{\theta}}_N) + \mathbf{H}_N \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N) \\ &\approx \mathbf{H}_N \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N),\end{aligned} \tag{30}$$

due to optimality of $\hat{\boldsymbol{\theta}}_N$, and where

$$\mathbf{H}_N = \left.\frac{\mathrm{d}^2 J_N(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_N}. \tag{31}$$

Substituting (30) into (28) gives

$$\begin{aligned}\dot{\mathbf{J}}_{N+1}(\boldsymbol{\theta}) \approx &\frac{1}{2}\mathbf{H}_N \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N) \\ &+ \sum_{\tau=0}^{\tau_{\max}} \{\boldsymbol{\psi}_N(r_{\hat{\boldsymbol{\theta}}_N} + \boldsymbol{\psi}_N^T \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_N) - 2\beta\rho) \\ &+ (1-2\alpha)\hat{r}_N \boldsymbol{\psi}_N\}.\end{aligned} \tag{32}$$

The fact that $\dot{\mathbf{J}}_{N+1}(\hat{\boldsymbol{\theta}}_{N+1}) = \mathbf{0}$, due to optimality of $\hat{\boldsymbol{\theta}}_{N+1}$, and some algebraic manipulations, give

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{N+1} \approx \hat{\boldsymbol{\theta}}_N - &\left(\frac{1}{2}\mathbf{H}_N + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}\right)^{-1} \\ &\cdot \sum_{\tau=0}^{\tau_{\max}}\{(r_{\hat{\boldsymbol{\theta}}_N} - 2\beta\rho + (1-2\alpha)\hat{r}_N)\boldsymbol{\psi}_N\}.\end{aligned} \tag{33}$$

The relation (33) constitutes a sequential update of the estimate $\hat{\boldsymbol{\theta}}_N$ to $\hat{\boldsymbol{\theta}}_{N+1}$, when a new data point $y(N+1)$ is added. However, a computationally more efficient form is obtained if $\mathbf{H}_N$ can be updated recursively. Differentiation of (28) with respect to $\boldsymbol{\theta}$ gives

$$\ddot{\mathbf{J}}_{N+1}(\boldsymbol{\theta}) \approx \frac{1}{2}\ddot{\mathbf{J}}_N(\boldsymbol{\theta}) + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}, \tag{34}$$

so

$$\mathbf{H}_{N+1} \approx \frac{1}{2}\mathbf{H}_N + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}. \tag{35}$$

Let

$$\mathbf{P}_N^{-1} = \mathbf{H}_{N+1}. \tag{36}$$

It then holds that

$$\mathbf{P}_N^{-1} = \frac{1}{2}\mathbf{P}_{N-1}^{-1} + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}. \tag{37}$$

The matrix inversion lemma

$$(\mathbf{D}+\mathbf{EF})^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{E}(\mathbf{I}+\mathbf{FD}^{-1}\mathbf{E})^{-1}\mathbf{FD}^{-1}, \tag{38}$$

where $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ and the identity matrix $\mathbf{I}$ have appropriate dimensions, then gives

$$\begin{aligned}\mathbf{P}_N &= \left(\frac{1}{2}\mathbf{P}_{N-1}^{-1} + \sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}\right)^{-1} \\ &= 2\mathbf{P}_{N-1} - \frac{4\mathbf{P}_{N-1}\sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N\boldsymbol{\psi}_N^T\}\mathbf{P}_{N-1}}{1+2\sum_{\tau=0}^{\tau_{\max}}\{\boldsymbol{\psi}_N^T\mathbf{P}_{N-1}\boldsymbol{\psi}_N\}}.\end{aligned} \tag{39}$$

The sequential estimation algorithm is now summarized as

$$\hat{\boldsymbol{\theta}}_{N+1} = \hat{\boldsymbol{\theta}}_N - \mathbf{P}_N\sum_{\tau=0}^{\tau_{\max}}\{(r_{\hat{\boldsymbol{\theta}}_N} - 2\beta\rho + (1-2\alpha)\hat{r}_N)\boldsymbol{\psi}_N\}, \tag{40}$$

where

- $\mathbf{P}_N$ is given iteratively by (39),
- $r_{\hat{\boldsymbol{\theta}}_N}$ is evaluated using (16),
- α, β and ρ are defined in (19), (20) and (21), respectively,
- $\hat{r}_N$ is given iteratively by (18),
- $\boldsymbol{\psi}_N$ is defined in (26).

This completes the derivation of the sequential estimation algorithm.

4. SAMPLING

The generation of discrete-time data from continuous-time ARMA processes is briefly described in this section, see, e.g., [14] for further details. The results presented here are used when generating data for the numerical study in the next section.

Let the sampling instants be $t = h, 2h, \ldots$. Integration of the process (6) over one sampling period gives

$$\begin{aligned}\mathbf{x}(kh+h) &= e^{\mathbf{A}h}\mathbf{x}(kh) + \int_{kh}^{kh+h} e^{\mathbf{A}(kh+h-s)}\mathbf{B}e(s)\mathrm{d}s \\ &\stackrel{\text{def}}{=} \mathbf{A}_d\mathbf{x}(kh) + \mathbf{v}(kh),\end{aligned} \tag{41}$$

with k being an integer. The random variable $\mathbf{v}(kh)$ is discrete-time white noise with covariance matrix

$$\mathbf{R}_{\mathbf{v}} = \int_0^h e^{\mathbf{A}s}\sigma_e^2\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T s}\mathrm{d}s. \tag{42}$$

The model (41) is therefore a standard discrete-time stochastic state space model, and it has the same covariance function, measured at multiples of the sampling interval, as the original continuous-time process (6). It also holds that the discrete-time spectral density tends to the continuous-time spectral density as the sampling interval tends to zero.

5. NUMERICAL STUDY

Consider the second order continuous-time AR process

$$(p+\ell_1)(p+\ell_2)y(t) = e(t), \tag{43}$$
$$\mathrm{E}\{e(t)e(s)\} = \delta(t-s), \tag{44}$$

where $\ell_1 \neq \ell_2$, $\ell_1 > 0$, $\ell_2 > 0$. By representing the process on state space form and proceeding as described in Section 2, the covariance function is given as

$$r_{\boldsymbol{\theta}}(\tau) = \frac{\ell_1 e^{-\ell_2\tau} - \ell_2 e^{-\ell_1\tau}}{2\ell_1\ell_2(\ell_1^2 - \ell_2^2)}, \tag{45}$$

where

$$\boldsymbol{\theta} = \begin{bmatrix}\ell_1, \ell_2\end{bmatrix}^T. \tag{46}$$

The true parameters are chosen as

$$\boldsymbol{\theta}_{\text{true}} = \begin{bmatrix}2, 3\end{bmatrix}^T \tag{47}$$

and discrete-time data are generated by sampling the process (43), as described in Section 4, with sampling interval $h = 0.1$.

The sequential algorithm (40) is used for estimating $\boldsymbol{\theta}$ in (46), with the initial parameters taken as

$$\boldsymbol{\theta}_{\text{init}} = \begin{bmatrix}1, 4\end{bmatrix}^T. \tag{48}$$

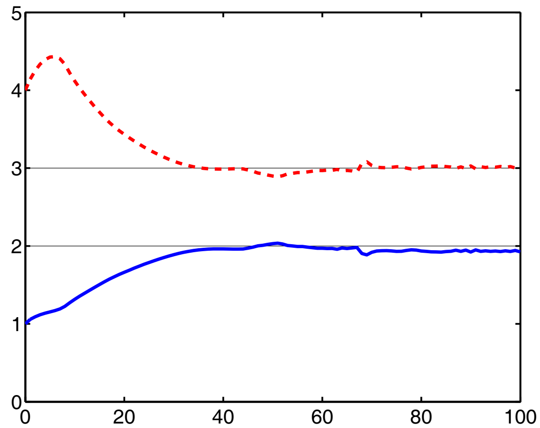

Fig. 1. The sequential parameter estimates of l_1 (solid) and l_2 (dashdot) as functions of the number of samples, i.e., the number of algorithm iterations. The true parameter values are indicated with solid lines.

Here, an initial value of $\hat{r}_N(\tau)$ is estimated using some of the data generated from the process. This is done before the sequential algorithm is started up with fresh data, i.e., data that have not been used for estimating $\hat{r}_N(\tau)$. An initial value of $\mathbf{P}_N$ is here given from (36) and (31), where (31) is computed numerically, and evaluated for $\boldsymbol{\theta}_{\text{init}}$. Moreover, the parameter $\tau_{\max}$ is chosen equal to four. The resulting sequential parameter estimates for a realization as described above are shown in Fig. 1, where it is seen that the estimates reach their true values after about 30 iterations.

6. CONCLUSIONS

An algorithm for sequential estimation of stochastic continuous-time parameters from discrete-time data was presented. The algorithm is based on fitting the covariance function of the process, which is parameterized by the signal parameters, to sample covariances. The continuous-time parameterization is therefore kept throughout the whole estimation procedure, and there is no need for transformations between continuous- and discrete-time parameterizations. Such transformations are known to be difficult, since a closed-form expression for the mapping of zeros does not exist. Another advantage with the sequential estimation algorithm proposed in the paper is that no signal derivatives have to be constructed from the discrete-time data. Instead, the information in the discrete-time data are utilized by means of covariances, which are possible to estimate with high accuracy. The sequential algorithm was given in compact expressions that are easy to implement, and the applicability was illustrated in a numerical study.

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