DELAYED EXPONENTIAL FITTING BY BEST TENSOR RANK-(R_1, R_2, R_3) APPROXIMATION

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ABSTRACT

We present a subspace-based scheme for the estimation of the poles (angular-frequencies and damping-factors) of a sum of damped and delayed sinusoids. In our model each component is supported over a different time frame, depending on the delay parameter. Classical subspace based methods are not suited to handle signals with varying time-supports. In this contribution, we propose a solution based on the best rank-(R_1, R_2, R_3) approximation of a partially structured Hankel tensor on which the data are mapped. We show, by means of an example, that our approach outperforms the current tensor and matrix-based approaches in terms of the accuracy of the damping parameter estimates.

1. INTRODUCTION

Estimation of the poles of a sum of windowed sinusoidal components is a key problem in harmonic retrieval [1], audio signal compression [2] and biomedical signal processing [3]. Among the numerous methods that have been proposed, the "subspace" methods, based on the invariance property of the signal subspace, form an important class.

Classically, these methods are used for the model-parameter estimation of a sum of Exponentially Damped Sinusoids (EDS) with the same time-support. Each component has the same length, namely, the length of the analysis window. In this contribution, we propose to use a more sophisticated model, called the *Partial* Damped and Delayed Sinusoidal model (PDDS). In this model, we add time-delay parameters that allow to time-shift each burst of EDS components. This modification is useful for the compact modeling of fast-time varying signals. For instance, [2] contains an application example in the context of audio transient compression.

Recently, multilinear algebra based variants of subspace methods have been derived. In [4] the Higher-Order Singular Value Decomposition (HOSVD), discussed in [5, 6], and the best rank-(R_1, R_2, R_3) approximation, discussed in [7, 6, 8], are used for the estimation of EDS from single-channel or multi-channel measurements. In [9] the HOSVD is used for PDDS modelling. In this paper we will use the best rank-(R_1, R_2, R_3) approximation for PDDS modelling. The tensor approximation will be computed by means of the Higher-Order Orthogonal Iteration (HOOI) algorithm [7, 6, 8], which is a tensor generalization of the well-known Orthogonal Iteration algorithm for the computation of the dominant subspace of a matrix.

The estimation of the damping-factors is known to be a difficult problem [1]. By means of an example, we will show that our approach is more accurate than the best current subspace methods.

2. THE PDDS MODEL AND ITS MULTICHANNEL STRUCTURE

2.1. Definition of the model

We define the complex M_k-PDDS model for $n \in [0 : N-1]$, by

$$\hat{s}_k(n) \triangleq \sum_{m=1}^{M_k} \alpha_{m,k} \,.\, z_{m,k}^{n-t_k} \,.\, \psi(n-t_k) \qquad (1)$$

where $\alpha_{m,k} = a_{m,k} e^{i\phi_{m,k}}$ is the complex amplitude, with $a_{m,k}$ and $\phi_{m,k}$ respectively the m-th real amplitude and initial phase of the k-th PDDS model of order M_k. $z_{m,k} = e^{d_{m,k}+i\omega_{m,k}}$ is the pole, with $d_{m,k}$ the (negative) damping factor and $\omega_{m,k}$ the angular-frequency. We denote by $\{t_k\}_{k\in[0:K]}$ the delay parameter set with $t_0 = 0$, $t_K = N-1$, $0 \le t_k < t_{k+1} \le N-1$ and $B_k = t_{k+1} - t_k$. The Heaviside function $\psi(n)$ is equal to "1" for $n \in [0 : N-1]$ and "0" otherwise. Note that there is a unique delay t_k for a set (sum) of M_k EDS waveforms. The M-PDDS model, where $M = \sum_{k=0}^{K-1} M_k$, is $\hat{s}(n) = \sum_{k=0}^{K-1} \hat{s}_k(n)$.

2.2. Channel with interference

We assume that the set of time-delays $\{t_k\}_{k\in[0:K]}$ is known. We can for example use a time-delay detector-estimator based on the variation of the energy of the time-envelope [2]. Our derivation now starts from the following observation:

The K signals

$$\{\hat{x}_k(n),\ n \in [0 : B_k - 1]\}_{k\in[0:K-1]}, \qquad (2)$$

given by

$$\hat{x}_k(n) \triangleq \underbrace{\hat{s}_k(n+t_k)}_{M_k\text{-EDS}} + \underbrace{\sum_{\ell=0}^{k-1}\sum_{m=1}^{M_\ell} \alpha_{m,\ell}\, z_{m,\ell}^{\sum_{u=\ell}^{k-1} B_u}\, z_{m,\ell}^{n}}_{\text{"interference"}}. \qquad (3)$$

are EDS models of order $\sum_{\ell=0}^{k} M_\ell$ with varying time-supports that can be seen as the sum of (1) a M_k-EDS signal $\hat{s}_k(n+t_k)$ and (2) an interfering attenuated $\left(\sum_{\ell=0}^{k-1} M_\ell\right)$-EDS model.

Each signal $\hat{x}_k(n)$ will be considered as a separate channel of B_k samples. Depending on the application, B_k may be small. Because the Fourier resolution is $O(B_k^{-1})$, Fourier analysis may not allow for an accurate estimation of the model parameters. Instead, we propose a subspace approach for the estimation of the set $\{z_{m,k}\}_{m\in[1:M_k]}$ from the set of channels (2).

However, the fact that the channels have a variable length poses a serious problem for the joint estimation of *all* the poles, *i.e.* $\{\{z_{m,k}\}_{m\in[1:M_k]},\ k \in [0:K-1]\}$. Indeed, subspace-based methods are not well suited to handle signals with varying time-supports. In the next section, we present a solution.

3. THE TENSOR APPROACH

In [9], we have compared several subspace methods in terms of estimation accuracy with which the parameters of a noisy fast time-varying signal are estimated. Here, we recall the best alternative.

3.1. Block-Vandermonde decomposition of a structured tensor

Consider a $B^- \times (B^+ - B^-) \times K$ partially structured Hankel-type tensor $\boldsymbol{\mathcal{A}}$. This three-way array can be interpreted as a series of "slabs" indexed by the channel index. More precisely, we have for the k-th slab:

$$[\boldsymbol{\mathcal{A}}]_k \stackrel{\triangle}{=} \mathcal{H}(\hat{\boldsymbol{x}}_k)\, \boldsymbol{W}_k \tag{4}$$

where we define $\mathcal{H}(.)$, the linear Hankel operator, as a mapping of a B_k-sample vector $\hat{\boldsymbol{x}}_k$ to a $B^- \times (B_k - B^-)$ matrix $\mathcal{H}(\hat{\boldsymbol{x}}_k)$ having constant skew diagonals. More precisely, every matrix $\mathcal{H}(\hat{\boldsymbol{x}}_k)$, represents the Hankel data matrix of the k-th channel of sample size B_k. Finally, we introduce the $(B_k - B^-) \times (B^+ - B^-)$ weighting matrix

$$\boldsymbol{W}_k \stackrel{\triangle}{=} \begin{bmatrix} \boldsymbol{I}_{B_k - B^-} & \boldsymbol{0}_{(B_k-B^-)\times(B^+-B_k)} \end{bmatrix} \tag{5}$$

with $B^+ \stackrel{\triangle}{=} \max_k B_k$ and $B^- \stackrel{\triangle}{=} [\delta \min_k B_k]$, where $[.]$ is the integer part of its argument. B^- (*i.e.* δ) is chosen such that $\forall k,\ B^- < B_k \leq B^+$.

Theorem 3.1 *If all the poles in the PDDS model are distinct and if we assume that*

$$M \quad < \quad \min(B^-, B^+ - B^-) \tag{6}$$

then tensor $\boldsymbol{\mathcal{A}}$ is a rank-(M, M, K) tensor that admits the following decomposition:

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{C}} \times_1 \boldsymbol{\Theta}. \tag{7}$$

Proof: In equation (7), the mode-1 product $\times_1$ [11] means that, for the k-th slab ($k \in [0:K-1]$ and see figure 1):

$$[\boldsymbol{\mathcal{A}}]_k = \boldsymbol{\Theta}\ [\boldsymbol{\mathcal{C}}]_k\,. \tag{8}$$

The matrix $[\boldsymbol{\mathcal{C}}]_k$ is $M \times (B^+ - B^-)$ and consists of K blocks of dimension $M_\ell \times (B^+ - B^-)$. Each block, indexed by ℓ, is given by

$$\begin{cases} \boldsymbol{\Psi}_\ell\, \boldsymbol{\Delta}_\ell^{\sum_{u=\ell}^{k-1} B_u}\, \boldsymbol{Z}_\ell^{(B_k-B^-)^T} \boldsymbol{W}_k & , \quad \text{for } \ell \in [0:k] \\ \boldsymbol{0}_{M_\ell \times M_\ell} & , \quad \text{for } \ell \in]k:K-1] \end{cases}$$

with $\boldsymbol{\Delta}_k = \text{diag}\{z_{1,k}, \ldots, z_{M_k,k}\}$ and $\boldsymbol{\Psi}_k = \text{diag}\{\alpha_{1,k}, \ldots, \alpha_{M_k,k}\}$ and, for $a \in [0:A-1]$, $[\boldsymbol{Z}_\ell^{(A)}]_{am} = z_{m,\ell}^a$ defining an $A \times M_\ell$ Vandermonde matrix. The matrix $\boldsymbol{\Theta}$ is defined as

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{Z}_0^{(B^-)} & \boldsymbol{Z}_1^{(B^-)} & \ldots & \boldsymbol{Z}_{K-1}^{(B^-)} \end{bmatrix}. \tag{9}$$

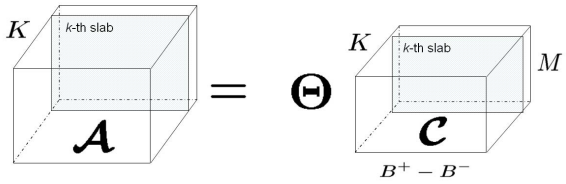

Fig. 1. Decomposition of tensor $\boldsymbol{\mathcal{A}}$.

First, note that the right multiplication by $\boldsymbol{W}_k$ in expression (4) leaves unchanged the column space of $\mathcal{H}(\hat{\boldsymbol{x}}_k)$, *ie.* $\mathcal{R}([\boldsymbol{\mathcal{A}}]_k) = \mathcal{R}(\mathcal{H}(\hat{\boldsymbol{x}}_k))$. Equation (7) follows from (3) by straightforward algebraic manipulations.

It remains to be shown that tensor $\boldsymbol{\mathcal{A}}$ is rank-(M, M, K). To this end, we introduce three so-called "matrix unfoldings" of $\boldsymbol{\mathcal{A}}$:

$$\boldsymbol{A}_{(1)} \quad : \quad B^- \times K(B^+ - B^-) \tag{10}$$

$$\boldsymbol{A}_{(2)} \quad : \quad (B^+ - B^-) \times KB^- \tag{11}$$

$$\boldsymbol{A}_{(3)} \quad : \quad K \times B^-(B^+ - B^-). \tag{12}$$

These matrices are different rearrangements of the values of the tensor. In matrix $\boldsymbol{A}_{(1)}$ all the column vectors of $\boldsymbol{\mathcal{A}}$ are stacked in a certain order. The row vectors and "three-mode vectors" of $\boldsymbol{\mathcal{A}}$ are stacked in $\boldsymbol{A}_{(2)}$ and $\boldsymbol{A}_{(3)}$, respectively. See [5, 6] for an exact definition. Tensor $\boldsymbol{\mathcal{A}}$ is by definition rank-(R_1, R_2, R_3) when the rank of matrix $\boldsymbol{A}_{(j)}$ is equal to R_j, $j \in [1:3]$. Due to the Hankel-type structure of tensor $\boldsymbol{\mathcal{A}}$, the matrix unfoldings are also structured. Consequently, if condition (6) is verified, then $\boldsymbol{A}_{(1)}$ and $\boldsymbol{A}_{(2)}$ are rank-M deficient matrices, thus $R_1 = R_2 = M$. As far as $\boldsymbol{A}_{(3)}$ is concerned, we have $K \leq M$ because $\forall k,\ M_k \neq 0$. In other words, the row dimension of $\boldsymbol{A}_{(3)}$ is smaller than the model order. Moreover, from condition (6) and $K \leq M$, follows that $K < B^-(B^+ - B^-)$. Hence, $\boldsymbol{A}_{(3)}$ is a "fat" matrix (row dimension lower than column dimension). This implies that $\boldsymbol{A}_{(3)}$ is not rank-deficient. We thus have $R_3 = K$. ■

3.2. Shift invariance of matrix $\boldsymbol{\Theta}$

Matrix $\boldsymbol{\Theta}$, defined in expression (9), is block-Vandermonde. Its shift invariance allows one to estimate all the poles of the PDDS model. Let $\boldsymbol{\Theta}_\downarrow$ and $\boldsymbol{\Theta}_\uparrow$ be the two submatrices of $\boldsymbol{\Theta}$, obtained by deleting the last and the first row, respectively. We then have

$$\boldsymbol{\Theta}_\downarrow\, \boldsymbol{\Delta} - \boldsymbol{\Theta}_\uparrow = \boldsymbol{0} \tag{13}$$

where $\boldsymbol{\Delta} = \text{diag}\,\{z_{m,k},\ m \in [1:M_k],\ k \in [0:K-1]\}$.

4. COMPUTATION OF THE SIGNAL SUBSPACE BY MEANS OF THE HOOI ALGORITHM

4.1. The matrix-pencil algorithm

The classical matrix-pencil algorithm [1] starts with the determination of a column-wise orthonormal matrix $\boldsymbol{U}$ that has the same column space as $\boldsymbol{\Theta}$. The columns of this matrix are given by the M dominant singular vectors of the Hankel data matrix. Next, define $\boldsymbol{U}_\downarrow$ and $\boldsymbol{U}_\uparrow$ in a similar way as $\boldsymbol{\Theta}_\downarrow$ and $\boldsymbol{\Theta}_\uparrow$ and compute $\boldsymbol{B}$ from

$$\boldsymbol{U}_\downarrow \, \boldsymbol{B} - \boldsymbol{U}_\uparrow = \boldsymbol{0}. \tag{14}$$

It can be shown that the matrix $\boldsymbol{B}$ is similar to $\boldsymbol{\Theta}$. In other words, the eigenvalues of $\boldsymbol{B}$ are the poles $z_{m,k},\ m \in [1 : M_k],\ k \in [0 : K-1]$.

For the estimation of the PDDS parameters, we are working in a multilinear context. It is natural to obtain a column-wise orthonormal matrix $\boldsymbol{U}$ from the HOSVD of $\boldsymbol{\mathcal{A}}$. This was done in [9]. However, as is shown in [7, 8], albeit the HOSVD yields a good rank-$(R_1, R_2, \ldots, R_J)$ approximation of a higher-order tensor, this approximation is usually not optimal. The optimal approximation can for instance be computed by means of the HOOI algorithm [7]. This indeed improves the signal subspace estimation, as will be shown in Section 5. In the two following subsections, we briefly present the HOSVD and the HOOI algorithm.

4.2. The Higher-Order Singular Value Decomposition

Theorem 4.1 (Jth-Order Singular Value Decomposition) *Every complex $(I_1 \times I_2 \times \ldots \times I_J)$-tensor $\boldsymbol{\mathcal{A}}$ can be written as the product*

$$\begin{aligned} \boldsymbol{\mathcal{A}} &= \boldsymbol{\mathcal{S}} \times_1 \boldsymbol{U}^{(1)} \times_2 \boldsymbol{U}^{(2)} \ldots \times_J \boldsymbol{U}^{(J)}, \quad \text{i.e.,} \quad (15) \\ (\boldsymbol{\mathcal{A}})_{i_1 i_2 \ldots i_J} &= \sum_{j_1 j_2 \cdots j_J} (\boldsymbol{\mathcal{S}})_{j_1 j_2 \ldots j_J} (\boldsymbol{U}^{(1)})_{i_1 j_1} \ldots (\boldsymbol{U}^{(J)})_{i_J j_J}, \end{aligned}$$

for all index values, in which $\boldsymbol{U}^{(j)}$ is a unitary $(I_j \times I_j)$-matrix and $\boldsymbol{\mathcal{S}}$ is an all-orthogonal and ordered complex $(I_1 \times I_2 \times \ldots \times I_J)$-tensor. All-orthogonality means that the subtensors $\boldsymbol{\mathcal{S}}_{i_j=\alpha}$, obtained by fixing the j-th index to α, are mutually orthogonal w.r.t. the standard inner product. Ordering means that $\left\|\boldsymbol{\mathcal{S}}_{i_j=1}\right\| \geqslant \left\|\boldsymbol{\mathcal{S}}_{i_j=2}\right\| \geqslant \ldots \geqslant \left\|\boldsymbol{\mathcal{S}}_{i_j=I_j}\right\| \geqslant 0$ for all possible values of j. The Frobenius-norms $\left\|\boldsymbol{\mathcal{S}}_{i_j=i}\right\|$, symbolized by $\sigma_i^{(j)}$, are j-mode singular values of $\boldsymbol{\mathcal{A}}$ and the vector $\boldsymbol{u}_i^{(j)}$ is an i-th j-mode singular vector.

This decomposition is a generalization of the matrix SVD because diagonality of the matrix containing the singular values, in the matrix case, is a special case of all-orthogonality. Also, the HOSVD of a second-order tensor (matrix) yields the matrix SVD, up to trivial indeterminacies. The matrix of j-mode singular vectors, $\boldsymbol{U}^{(j)}$, can be found as the matrix of left singular vectors of the matrix unfolding $\boldsymbol{A}_{(j)}$, defined in (10) – (12). The j-mode singular values correspond to the singular values of this matrix unfolding. The core tensor $\boldsymbol{\mathcal{S}}$ can then be computed by bringing the matrices of j-mode singular vectors to the left side of equation (15):

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{A}} \times_1 \boldsymbol{U}^{(1)^H} \times_2 \boldsymbol{U}^{(2)^H} \ldots \times_J \boldsymbol{U}^{(J)^H}. \tag{16}$$

For our data tensor $\boldsymbol{\mathcal{A}}$, it can be shown that, in the absence of noise, $\mathcal{R}(\boldsymbol{U}^{(1)}) = \mathcal{R}(\boldsymbol{\Theta})$. Hence, we can extract the poles from the eigen-decomposition of the product $\boldsymbol{U}_\downarrow^{(1)^\dagger} \boldsymbol{U}_\uparrow^{(1)}$, in which $\dagger$ denotes the Moore-Penrose pseudo-inverse.

4.3. Best rank-$(R_1, R_2, \ldots, R_J)$ approximation

In this section we consider a multilinear generalization of the best rank-R approximation of a given matrix. More precisely, given a J-th order tensor $\boldsymbol{\mathcal{A}} \in \mathbb{C}^{I_1 \times I_2 \times \ldots I_J}$, we want to find a tensor $\hat{\boldsymbol{\mathcal{A}}} \in \mathbb{C}^{I_1 \times I_2 \times \ldots I_J}$, with $\text{rank}(\hat{\boldsymbol{A}}_{(1)}) = R_1$, $\text{rank}(\hat{\boldsymbol{A}}_{(2)}) = R_2$, …, $\text{rank}(\hat{\boldsymbol{A}}_{(J)}) = R_J$, that minimizes the least-squares cost function

$$f(\hat{\boldsymbol{\mathcal{A}}}) = \|\boldsymbol{\mathcal{A}} - \hat{\boldsymbol{\mathcal{A}}}\|^2. \tag{17}$$

The rank conditions imply that $\hat{\boldsymbol{\mathcal{A}}}$ can be decomposed as

$$\hat{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{B}} \times_1 \boldsymbol{U}^{(1)} \times_2 \boldsymbol{U}^{(2)} \ldots \times_J \boldsymbol{U}^{(J)}, \tag{18}$$

in which $\boldsymbol{U}^{(1)} \in \mathbb{C}^{I_1 \times R_1}, \boldsymbol{U}^{(2)} \in \mathbb{C}^{I_2 \times R_2}, \ldots, \boldsymbol{U}^{(J)} \in \mathbb{C}^{I_J \times R_J}$ each have orthonormal columns and $\boldsymbol{\mathcal{B}} \in \mathbb{C}^{R_1 \times R_2 \times \ldots \times R_J}$.

Similarly to the second-order case, where the best approximation of a given matrix $\boldsymbol{A} \in \mathbb{C}^{I_1 \times I_2}$ by a matrix $\hat{\boldsymbol{A}} = \boldsymbol{U}^{(1)} \cdot \boldsymbol{B} \cdot \boldsymbol{U}^{(2)^H}$, with $\boldsymbol{U}^{(1)} \in \mathbb{C}^{I_1 \times R}$ and $\boldsymbol{U}^{(1)} \in \mathbb{C}^{I_2 \times R}$ column-wise orthonormal, is equivalent to the maximization of $\|\boldsymbol{U}^{(1)^H} \cdot \boldsymbol{A} \cdot \boldsymbol{U}^{(2)}\|$, we have that the minimization of $f(\hat{A})$ is equivalent to the maximization of

$$g(\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \ldots, \boldsymbol{U}^{(J)}) = \|\boldsymbol{\mathcal{A}} \times_1 \boldsymbol{U}^{(1)^H} \times_2 \boldsymbol{U}^{(2)^H} \ldots \times_J \boldsymbol{U}^{(J)^H}\|^2.$$

The optimal core tensor follows from

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{A}} \times_1 \boldsymbol{U}^{(1)^H} \times_2 \boldsymbol{U}^{(2)^H} \ldots \times_J \boldsymbol{U}^{(J)^H}. \tag{19}$$

It is natural to question whether the best rank-$(R_1, R_2, \ldots, R_J)$ approximation of a higher-order tensor can be obtained by truncation of the HOSVD, in analogy with the matrix case. The situation turns out to be quite different for tensors [7, 8]. By discarding the smallest n-mode singular values, one obtains a tensor $\hat{\boldsymbol{\mathcal{A}}}$ that is in general not the best possible approximation under the given n-mode rank constraints. In our application, the truncated HOSVD and the best rank-$(R_1, R_2, \ldots, R_J)$ approximation are generically only equal in the absence of noise.

In [7, 8] the following approach was followed for the computation of the best rank-$(R_1, R_2, \ldots, R_J)$ approximation. Imagine that the matrices $\boldsymbol{U}^{(1)}, \ldots, \boldsymbol{U}^{(j-1)}, \boldsymbol{U}^{(j+1)}, \ldots, \boldsymbol{U}^{(J)}$ are fixed and that the only unknown is the column-wise orthonormal matrix $\boldsymbol{U}^{(j)}$. We have

$$g = \|\tilde{\boldsymbol{\mathcal{A}}}^{(j)} \times_j \boldsymbol{U}^{(j)^H}\|^2, \tag{20}$$

in which $\tilde{\boldsymbol{\mathcal{A}}}^{(j)} \triangleq \boldsymbol{\mathcal{A}} \times_1 \boldsymbol{U}^{(1)^H} \ldots \times_{j-1} \boldsymbol{U}^{(j-1)^H} \times_{j+1} \boldsymbol{U}^{(j+1)^H} \ldots \times_J \boldsymbol{U}^{(J)^H}$. Hence the columns of $\boldsymbol{U}^{(j)}$ can be found as an orthonormal basis for the dominant subspace of $\tilde{\boldsymbol{\mathcal{A}}}^{(j)}$. Repeating this procedure for different mode numbers leads to an Alternating Least Squares (ALS) algorithm for the (local) maximization of $f(\hat{\boldsymbol{\mathcal{A}}})$: in each step the estimate of one of the matrices $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \ldots, \boldsymbol{U}^{(J)}$ is optimized, while the other matrix estimates are kept constant.

It makes sense to initialize the HOOI with the truncated HOSVD. The HOSVD-estimate usually belongs to the attraction region of the best rank-$(R_1, R_2, \ldots, R_J)$ approximation, although there is no absolute guarantee of convergence to the global optimum [7].

5. SIMULATIONS

Consider the noisy synthetic signal of figure 2 given by $\boldsymbol{s} = \hat{\boldsymbol{s}} + \sigma\boldsymbol{w}$. $\sigma\boldsymbol{w}$ denotes a white Gaussian perturbation with variance σ^2. $\hat{\boldsymbol{s}}$ is a 200-sample 2-PDDS signal. The performance criterion is the Normalized Mean Square Error (NMSE), in logarithmic scale, evaluated for several Signal-to-Noise Ratios (SNRs) and averaged over 500 trials. The NMSE is defined by the mean ratio of the square difference between the true parameter value and its estimate over the square value of the true parameter. The SNR in dB is defined by $\text{SNR}(\hat{\boldsymbol{s}}, \sigma\boldsymbol{w}) = 10\log_{10}(||\hat{\boldsymbol{s}}||_2^2/\sigma^2)$.

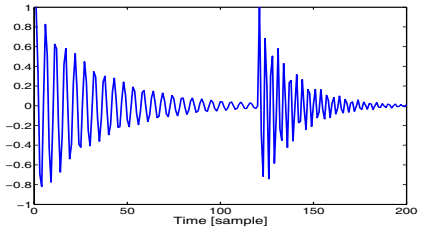

Fig. 2. Test signals $\Re e\{\hat{\boldsymbol{s}}\}$.

In figure 3, we show the estimation errors for the angular-frequencies ($\hat{\omega}_{1,0}$ and $\hat{\omega}_{1,1}$) and the damping factors ($\hat{d}_{1,0}$ and $\hat{d}_{1,1}$) as obtained by the T1 algorithm (tensor method based on the HOSVD), by the T1+ algorithm (tensor method based on the HOOI) and the MC1 algorithm (currently the best matrix approach [2]). The latter method is based on the decomposition of matrix $[\mathcal{H}(\hat{\boldsymbol{x}}_0) \ \ldots \ \mathcal{H}(\hat{\boldsymbol{x}}_{K-1})]$. We show also the Conditional Cramer-Rao Bound (CCRB) for the PDDS model. This bound is derived in reference [10]. Note that the term *conditional* means that this bound is computed with the exact knowledge of the discrete time-delay parameters.

As can be seen in figure 3, method T1 is equivalent to the best matrix approach, MC1. Further, we conclude that the three algorithms perform similarly w.r.t. the angular-frequencies. On the other hand, the T1+ algorithm is more robust to noise than the T1 and MC1 algorithms as far as the estimation of the damping factors is concerned. In particular, we observe results quite close to the optimal CCRB.

6. CONCLUSION

In this paper, we have presented a subspace-based method for estimation of the poles (angular-frequencies and damping-factors) of damped and delayed sinusoids, having different time-supports. The algorithm uses multilinear algebraic tools, applied to a structured data tensor. Fitting a synthetic transient signal showed that our approach outperforms the current tensor and matrix methods for the estimation of PDDS model parameters.

Acknowledgements

Part of this work is supported by (1) the Research Council K.U.Leuven: Concerted Research Action GOA-MEFISTO-666, (2) the Flemish Government: the F.W.O. project G.0240.99 and the F.W.O. Research Communities ICCoS and ANMMM, and (3) the Belgian Federal Government: Interuniversity Poles of Attraction Programme IUAP V-22.

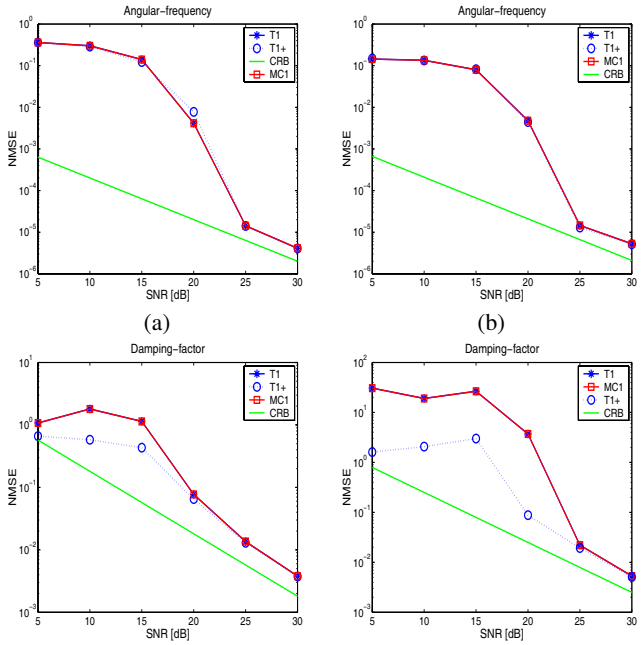

Fig. 3. (a) NMSE($\hat{\omega}_{1,0}$), (b) NMSE($\hat{\omega}_{1,1}$), (c) NMSE($\hat{d}_{1,0}$), (d) NMSE($\hat{d}_{1,1}$).

7. REFERENCES

[1] Y. Hua and T.K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise", *IEEE Trans. on Acoustic, Speech and Signal Processing*, Vol. 38 Issue: 5 , May 1990.

[2] R. Boyer and K. Abed-Meraim, "Audio Modeling based on Delayed Sinusoids", *IEEE Trans on Speech and Audio Processing*, Vol. 12, No. 2, March 2004.

[3] S. Van Huffel, C. Decanniere, H. Chen, P. Van Hecke, "Algorithm for time-domain NMR data fitting based on total least squares", *Journal of Magnetic Resonance A*, Vol. 110, 1994.

[4] J.-M. Papy, L. De Lathauwer, S. Van Huffel, "Exponential Data Fitting Using Multilinear Algebra. The Single-Channel and Multichannel Case." *Num. Lin. Alg. Appl.*, submitted (preprint available at http://www.esat.kuleuven.ac.be/ sistawww/cgi-bin/pub.pl).

[5] L. De Lathauwer, B. De Moor, J. Vandewalle, "A Multilinear Singular Value Decomposition", *SIAM J. Matrix Anal. Appl.*, Vol. 21, No. 4, April 2000, pp. 1253–1278.

[6] L. De Lathauwer, J. Vandewalle, "Dimensionality Reduction in Higher-Order Signal Processing and Rank-$(R_1, R_2, \ldots, R_N)$ Reduction in Multilinear Algebra", *Lin. Alg. Appl.*, to appear.

[7] L. De Lathauwer, B. De Moor, J. Vandewalle, "On the Best Rank-1 and Rank-$(R_1, R_2, \ldots, R_N)$ Approximation of Higher-Order Tensors", *SIAM J. Matrix Anal. Appl.*, Vol. 21, No. 4, April 2000, pp. 1324–1342.

[8] P.M Kroonenberg, *Three-Mode Principal Component Analysis*, Leiden: DSWO Press.

[9] R. Boyer and K. Abed-Meraim, "Structured tensor based-algorithm for delayed exponential fitting", *Proc of Asilomar Conference on Signals, Systems and Computers*, 2004. .

[10] R. Boyer and K. Abed-Meraim, "Damped and Delayed model for Transient Modeling", *IEEE Trans. on Signal Processing*, to appear.

[11] L.R. Tucker, Some Mathematical Notes on Three-mode Factor Analysis, *Psychometrika*, Vol. 31, 1966, pp. 279-311.