Staying Rich: LTI Systems Which Preserve Signal Richness

P. P. Vaidyanathan and Borching Su
Dept. of Electrical Engineering, California Institute of Technology
Pasadena, CA 91125, USA
ppvnath@systems.caltech.edu borching@caltech.edu

Abstract. There are many ways to define richness of a discrete time signal. This paper considers a particular definition and explores the conditions under which a linear time invariant system preserves the richness property. Several examples are presented to clarify the issues involved in the problem. Some sufficient conditions are presented. Also presented are necessary and sufficient conditions for some special cases. A set of necessary and sufficient conditions for the most general case is not known at this time.[1]

1. INTRODUCTION

Signals are often considered to be "rich" if they satisfy certain fullness properties appropriate for an application under discussion. In some applications a signal is regarded as rich if it has nonzero energy at all frequencies. In some applications a sequence of $M \times 1$ vectors $\mathbf{x}(n), n \geq 0$ is defined to be **rich** or **rank-rich** if the matrix

$$[\,\mathbf{x}(0) \quad \mathbf{x}(1) \quad \ldots \quad \mathbf{x}(K_x)\,]$$

has rank M for sufficiently large K_x. This property is important, for example, when we try to identify an unknown communication channel from output measurements alone using filter bank precoders [4]. Now, signals are sometimes preconditioned by linear transformations before they are used in such an application. An example is the case where antipodal paraunitary matrices [3] are used to precondition the signal before being sent through the filter bank precoder. The advantages of the use of such antipodal preconditioners are explained in detail in [3].

In these and other applications an interesting theoretical question that comes up is this: if a rich signal is input to a linear time invariant (LTI) system, then does the output continue to have the richness property? This fundamental question, rather than the applications, is the focus of this paper. Let the linear time invariant system be characterized by the $M \times M$ transfer matrix

$$\mathbf{H}(z) = \sum_n \mathbf{h}(n) z^{-n}$$

as shown in Fig. 1 so that

$$\mathbf{y}(n) = \sum_k \mathbf{h}(k)\mathbf{x}(n-k)$$

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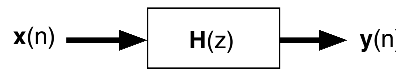

Fig. 1. A multi-input multi-output LTI system.

If the system $\mathbf{H}(z)$ is such that rank-rich inputs always produces rank-rich outputs we say that $\mathbf{H}(z)$ is richness-preserving (RP). Some sufficient conditions for this will be presented in Sec. 3. Also presented are necessary and sufficient conditions for some special cases (Sec. 4.) A set of necessary and sufficient conditions for the most general case are not known to the authors at the time of this writing. Throughout the paper we will use the term richness to imply rank-richness.

2. FORMULATION AND EXAMPLES

Assume that the system is causal so that the input $\mathbf{x}(n), n \geq 0$ produces a causal output. This output is rich if there exists an integer K_y such that

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \ldots \quad \mathbf{y}(K_y)\,]$$

has rank M. We assume for simplicity that $\mathbf{H}(z)$ is causal FIR with order N. Then $\mathcal{Y} = \mathcal{H}\mathcal{X}$ where

$$\mathcal{Y} = [\mathbf{y}(0)\;\mathbf{y}(1)\;\ldots\;\mathbf{y}(K_y)], \quad \mathcal{H} = [\mathbf{h}(0)\;\mathbf{h}(1)\;\ldots\;\mathbf{h}(N)]$$

and

$$\mathcal{X} = \begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}(2) & \ldots & \mathbf{x}(K_y) \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) & \ldots & \mathbf{x}(K_y-1) \\ \mathbf{0} & \mathbf{0} & \mathbf{x}(0) & \ldots & \mathbf{x}(K_y-2) \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{x}(K_y-N) \end{bmatrix}$$

The matrix $\mathcal{H}$ has size $M \times M(N+1)$. With ρ_y, ρ_h and ρ_x denoting the ranks of $\mathcal{Y}, \mathcal{H}$ and $\mathcal{X}$ respectively, we have from Sylvester's inequality [2]:

$$\rho_h + \rho_x - M(N+1) \leq \rho_y \leq \min\,(\rho_h, \rho_x) \qquad (1)$$

Observe that if the output matrix $\mathcal{Y}$ has to have rank M it is *necessary* that the filter matrix $\mathcal{H}$ have rank M. For example, if one of the $\mathbf{h}(n)$'s has rank M, this is satisfied. We will produce examples to demonstrate that this necessary condition is in fact not sufficient. In fact the examples

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also show that many stantard systems such as unimodular and paraunitary matrices do not preserve richness!

Example 1. To demonstrate that the rank-M property of the filter matrix $\mathcal{H}$ is not sufficient, consider the following example with $M = 2$:

$$\mathbf{H}(z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then

$$\mathcal{H} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix},$$

and has rank $M = 2$. Suppose the input signal is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

with $\mathbf{x}(n) = \mathbf{0}$ otherwise. Clearly this input is rich because $[\,\mathbf{x}(0) \quad \mathbf{x}(1)\,]$ has rank two. The output can have only three nonzero samples, so that the largest output matrix we need to look at is:

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \mathbf{y}(2)\,] = \underbrace{[\,\mathbf{h}(0) \quad \mathbf{h}(1)\,]}_{\mathcal{H}} \underbrace{\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{0} \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) \end{bmatrix}}_{\mathcal{X}}$$

We have

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \mathbf{y}(2)\,] = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

which shows that the output matrix has rank one. Thus, richness of the input is not preserved at the output even though the matrix $\mathcal{H}$ has full rank M. In this example $\mathbf{H}(z)$ happens to be a **paraunitary** matrix [5], that is, it satisfies $\mathbf{H}^{\dagger}(e^{j\omega})\mathbf{H}(e^{j\omega}) = c\mathbf{I}$, where superscript dagger denotes transpose conjugation. Thus paraunitary matrices do not necessarily preserve richness.

Example 2. Consider again $M = 2$ and let

$$\mathbf{H}(z) = \begin{bmatrix} 1 + z^{-1} & -z^{-1} \\ z^{-1} & 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Then

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix},$$

and has rank $M = 2$. Consider the input

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with $\mathbf{x}(n) = \mathbf{0}$ otherwise. Then the output matrix is

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \mathbf{y}(2)\,] = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank one. Again richness of the input is not preserved at the output, though $\mathcal{H}$ has full rank M. In this example $\mathbf{H}(z)$ happens to be a **unimodular** matrix [1], that is, it has determinant =1 so that its inverse is an FIR matrix as well. The example shows that unimodular matrices do not necessarily preserve richness.

Example 3. An enriching example. If the input to an LTI system is not rich, then is it at all possible for the output to be rich? The following example shows that this can happen. Let $M = 2$ and consider

$$\mathbf{H}(z) = \begin{bmatrix} 1 & z^{-1} \\ z^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Suppose we apply the input

$$\mathbf{x}(n) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(n)$$

This is evidently not rich. The output is

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $\mathbf{y}(n) = 0$ otherwise. Since

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1)\,] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has rank two, it follows that $\mathbf{y}(n)$ is rich. So this system can turn a nonrich input into a rich output. But this same system can also turn a rich input into a nonrich output as the next example shows. Thus let

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

with $\mathbf{x}(n) = 0$ otherwise. This is a rich input, and

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \mathbf{y}(2)\,] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This has rank one. Since $\mathbf{y}(n) = 0$ for all other n, this shows that the output is not rich.

A consequence of the preceding example is this: suppose we have a cascade of two systems such that the first system does not preserve richness. This does not prove that the cascade is not richness-preserving because it is possible that the second system can make up for it. A trivial example is a cascade of the unimodular matrix in Example 2 with its inverse (which is also causal and unimodular).

Since the product is identity it preserves richness. But at least one of the factors in the product is not a richness preserving system.

3. SUFFICIENT CONDITIONS

Even though we do not know of a necessary and sufficient condition on the system that preserves richness we can find some nontrivial sufficient conditions. For example suppose $\mathbf{H}(z)$ is a constant, that is, $\mathbf{H}(z) = \mathbf{A}$. Then the necessary condition that $\mathcal{H}$ have full rank implies that $\mathbf{A}$ should have full rank, and it turns out that this is also sufficient for richness. This follows by writing

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \ldots \quad \mathbf{y}(K_y)\,] = \mathbf{A}\,[\,\mathbf{x}(0) \quad \mathbf{x}(1) \quad \ldots \quad \mathbf{x}(K_y)\,]$$

If K_y is large enough such that $\mathcal{X}$ has rank M then $\mathcal{Y}$ indeed has rank M because $\mathbf{A}$ is nonsingular. Thus richness is trivially preserved. A more general family of LTI systems preserving richness is given next.

Theorem 1. Consider the Nth order FIR system

$$\mathbf{H}(z) = \mathbf{A}\Big(g_0 + g_1 z^{-1} + \ldots + g_N z^{-N}\Big)$$

where $\mathbf{A}$ is an $M \times M$ nonsingular matrix and $g_0 \neq 0$. This system preserves richness. $\diamondsuit$

Proof. Consider a rich input $\mathbf{x}(n)$ and write the output matrix $\mathcal{Y} = \mathcal{H}\mathcal{X}$ as

$$\underbrace{[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \ldots \quad \mathbf{y}(K_x)\,]}_{\mathcal{Y}} = \underbrace{[\,g_0\mathbf{A} \quad g_1\mathbf{A} \quad \ldots \quad g_N\mathbf{A}\,]}_{\mathcal{H}}$$

$$\times \underbrace{\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}(2) & \ldots & \mathbf{x}(K_x) \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) & \ldots & \mathbf{x}(K_x-1) \\ \mathbf{0} & \mathbf{0} & \mathbf{x}(0) & \ldots & \mathbf{x}(K_x-2) \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{x}(K_x-N) \end{bmatrix}}_{\mathcal{X}}$$

Let us assume that K_x is large enough so that

$$[\,\mathbf{x}(0) \quad \mathbf{x}(1) \quad \mathbf{x}(2) \quad \ldots \quad \mathbf{x}(K_x)\,] \tag{2}$$

has full rank M. Since $\mathbf{x}(n)$ is rich such a K_x exists. We now prove that $\mathbf{y}(n)$ is rich by proving that $\mathcal{Y}$ has rank M as well. Assume the contrary, that is, suppose there exists a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{v}^\dagger\,[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \ldots \quad \mathbf{y}(K_x)\,] = \mathbf{0},$$

that is,

$$[\,g_0\mathbf{v}^\dagger\mathbf{A} \quad g_1\mathbf{v}^\dagger\mathbf{A} \quad \ldots \quad g_N\mathbf{v}^\dagger\mathbf{A}\,]$$

$$\times \begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}(2) & \ldots & \mathbf{x}(K_x) \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) & \ldots & \mathbf{x}(K_x-1) \\ \mathbf{0} & \mathbf{0} & \mathbf{x}(0) & \ldots & \mathbf{x}(K_x-2) \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{x}(K_x-N) \end{bmatrix} = \mathbf{0}$$

From the 0th column of this equation, we have $\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(0) = \mathbf{0}$ because $g_0 \neq 0$. From the next column,

$$g_0\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(1) + g_1\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(0) = \mathbf{0}$$

which implies $\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(1) = \mathbf{0}$ because the second term is zero. From the next column, $g_0\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(2) + g_1\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(1) + g_2\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(0) = \mathbf{0}$ and since the last two terms are zero this implies $\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(2) = \mathbf{0}$. Proceeding like this, we see that $\mathbf{v}^\dagger\mathbf{A}\mathbf{x}(n) = \mathbf{0}$ for $0 \leq n \leq K_x$, that is,

$$\mathbf{v}^\dagger\mathbf{A}\,[\,\mathbf{x}(0) \quad \mathbf{x}(1) \quad \mathbf{x}(2) \quad \ldots \quad \mathbf{x}(K_x)\,] = \mathbf{0}$$

Since $\mathbf{A}$ is nonsingular, $\mathbf{v}^\dagger\mathbf{A} \neq \mathbf{0}$ and the preceding equation contradicts the full-rank property of (2). This proves the claim of the theorem. $\bigtriangledown\bigtriangledown\bigtriangledown$

It is easily argued that if $\mathbf{H}(z)$ satisfies the conditions of the Theorem, then a non-rich input cannot produced rich output. The reason is that the output samples $\mathbf{y}(n)$ are just linear combinations of $\mathbf{A}\mathbf{x}(n-k)$ so the space spanned by the output vectors $\{\mathbf{y}(n)\}$ cannot have dimension larger than the space spanned by the input vectors $\{\mathbf{x}(n)\}$.

The form shown in Theorem 1, though sufficient for richness-preservation, is *not necessary.* This is shown by the following example: let

$$\mathbf{H}(z) = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} + z^{-1}\begin{bmatrix} 0 & 0 \\ 1 & a \end{bmatrix} \tag{3}$$

This system is not in the form of Theorem 1, but it preserves richeness for any a.

Justification. Observe that

$$[\,\mathbf{y}(0) \quad \mathbf{y}(1) \quad \ldots\,] = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \end{bmatrix}\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \ldots \\ \mathbf{0} & \mathbf{x}(0) & \ldots \end{bmatrix}$$

We assume the input is rich but the output is not, and exhibit a contradiction. When the output is not rich, there exists a nonzero vector $[\,v_0 \quad v_1\,]$ which annihilates the preceding matrix from the left, that is,

$$[\,v_0 \quad v_1\,]\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \end{bmatrix}\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}(2) & \ldots \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) & \ldots \end{bmatrix} = \mathbf{0}$$

or equivalently

$$[\,v_0 \quad av_0 \quad v_1 \quad av_1\,]\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}(2) & \ldots \\ \mathbf{0} & \mathbf{x}(0) & \mathbf{x}(1) & \ldots \end{bmatrix} = \mathbf{0} \tag{4}$$

If v_0 or v_1 is zero this implies $[\,1 \quad a\,]\,[\,\mathbf{x}(0) \quad \mathbf{x}(1) \quad \ldots\,] = \mathbf{0}$ and contradicts the assumed richness of the input. So let us assume $v_0, v_1 \neq 0$. Eq. (4) yields a succession of equations:

$$v_0\,[\,1 \quad a\,]\,\mathbf{x}(n) + v_1\,[\,1 \quad a\,]\,\mathbf{x}(n-1) = 0, \quad n \geq 0$$

Note that the first equation simply says $[\,1 \quad a\,]\,\mathbf{x}(0) = \mathbf{0}$ because $\mathbf{x}(-1) = \mathbf{0}$. By substituting from the first equation into the second, and then into the third, and so forth, we conclude from this that $[1 \quad a]\,\mathbf{x}(n) = 0$ for all $n \geq 0$, which contradicts the assumed richness of the input. This concludes the proof that (3) preserves richness. $\bigtriangledown\bigtriangledown\bigtriangledown$

For the system shown in Eq. (3) if $\mathbf{x}(n) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(n)$ then $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which shows that the output is rich though the input is not. So the system is not only richeness-preserving, it can in fact enrich some nonrich signals!

4. A SPECIAL CASE

For the special case of first order 2×2 systems with nonsingular $\mathbf{h}(0)$, the form given in Theroem 1 is not only sufficient but necessary as well. More precisely, we have the following:

Theorem 2. Consider the first order FIR system

$$\mathbf{H}(z) = \mathbf{h}(0) + z^{-1}\mathbf{h}(1)$$

with size 2×2 and assume $\mathbf{h}(0)$ is nonsingular. Then $\mathbf{H}(z)$ preserves richness if and only if $\mathbf{h}(1) = \rho\mathbf{h}(0)$ for some scalar constant ρ. $\diamondsuit$

Proof. Since $\mathbf{h}(0)$ is nonsingular, we can write $\mathbf{H}(z) = \mathbf{h}(0)(\mathbf{I} + \mathbf{B}z^{-1})$. The nonsingular factor $\mathbf{h}(0)$ does not affect the rank of the output matrix. So $\mathbf{H}(z)$ is richness preserving if and only if $(\mathbf{I} + \mathbf{B}z^{-1})$, which has the form

$$\mathbf{G}(z) = \mathbf{I} + z^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

preserves richness. Let us explore the conditions for this. Consider the input

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} d \\ -c \end{bmatrix}$$

with $\mathbf{x}(n) = \mathbf{0}$ otherwise. This produces the output

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}(1) = \begin{bmatrix} a+d \\ 0 \end{bmatrix}, \quad \mathbf{y}(2) = \begin{bmatrix} ad-bc \\ 0 \end{bmatrix},$$

and $\mathbf{y}(n) = \mathbf{0}$ otherwise. We see that if $c \neq 0$ then the input is rich ($[\mathbf{x}(0) \quad \mathbf{x}(1)]$ has rank 2) but the output is not. So $c = 0$ is a necessary condition for richness preservation. A slight variation of this construction shows that $b = 0$ is necessary as well. Thus, in order to preserve richness $\mathbf{G}(z)$ has to be of the form

$$\mathbf{G}(z) = \mathbf{I} + z^{-1} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

If we now choose the input

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} d \\ 2a \end{bmatrix},$$

with $\mathbf{x}(n) = \mathbf{0}$ otherwise, then

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y}(1) = (a+d) \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y}(2) = ad \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

with $\mathbf{y}(n) = \mathbf{0}$ otherwise. If $a \neq d$, then the input is rich whereas the output is not. This shows that $a = d$ is a necessary condition. Thus $\mathbf{G} = \mathbf{I} + \rho\mathbf{I}z^{-1}$, so $\mathbf{h}(1) = \rho\mathbf{h}(0)$ indeed. $\bigtriangledown\bigtriangledown\bigtriangledown$

5. CONCLUDING REMARKS AND OPEN ISSUES

Under the definition of richness considered in this paper, it remains to find a set of necessary and sufficient conditions on the LTI system that preserves richness. The definition given in this paper is equivalent to the statement that $\mathbf{v}^\dagger\mathbf{X}(e^{j\omega})$ be not identically zero for all ω for a fixed nonzero vector $\mathbf{v}$. It is also of interest to consider variations in the definition. For example, a variation would be as follows: $\mathbf{x}(n)$ is rich if there is no frequency interval of the form $[\omega_0, \omega_0 + \epsilon]$ such that $\mathbf{v}^\dagger\mathbf{X}(e^{j\omega})$ is identically zero there for constant $\mathbf{v} \neq \mathbf{0}$. Another definition could be this: $\mathbf{x}(n)$ is rich if for any initial time n_0 there exists an integer K such that

$$[\mathbf{x}(n_0) \quad \mathbf{x}(n_0+1) \quad \ldots \quad \mathbf{x}(n_0+K)]$$

has rank M (with K not necessarily the same for all n_0). This appears to be a more practical definition for richness. It is clear that paraunitarity and unimodularity are still not sufficient for preservation of this richness. Thus consider Example 1 again. Redefine the input as the periodic signal

$$\mathbf{a}, \mathbf{b}, \mathbf{0}, \ldots \mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{0}, \ldots \mathbf{0}, \mathbf{a}, \mathbf{b}, \ldots$$

where $\mathbf{a} = \mathbf{x}(0)$ and $\mathbf{b} = \mathbf{x}(1)$ as in Example 1. With a sufficiently large block of zeros separating the nonzero input samples it is clear that the output is the periodic signal

$$\mathbf{0}, \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \mathbf{0}, \ldots \mathbf{0}, \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \mathbf{0}, \ldots \mathbf{0}, \ldots$$

Though the input is rich according to the revised definition, the output is not, because the rank of the output matrix $\mathcal{Y}$ can never exceed unity. If we adopt one of these definitions, and want a similar kind of richness to be preserved at the output, then what are the necessary and sufficient conditions on the LTI system? These are open questions and demand further investigation.

REFERENCES

[1] Gantmacher, F. R., *The theory of matrices*, vol. 1, Chelsea Publishing Co., N.Y., 1977.

[2] Horn, R. A., and Johnson, C. R *Matrix analysis*, Cambridge University Press, 1985.

[3] Phoong, S-M., Chang, K.-Y., and Lin, Y.-P, "Antipodal paraunitary precoding for OFDM application", Proc. IEEE Intl. Symp. on Circuits and Systems, Vancouver, Canada, May 2004.

[4] Scaglione, A., Giannakis, G. B., and Barbarossa, S. "Redundant filter bank precoders and equalizers Part II: Synchronization and direct equalization", IEEE Trans. Signal Processing, vol. 47, pp. 2007-2022, July 1999.

[5] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice Hall, Inc., 1993.