HYBRID WIENER FILTER

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ABSTRACT

Known forms of Wiener filter do not adequately address the discrete-time signal processing of continuous random processes. In this paper, we first formulate this problem and subsequently derive its solution. The crux of the work is the development of autocorrelation function invariant discrete equivalent of a continuous system. This in turn enables us to transform the hybrid Wiener filtering problem into a purely discrete one. An example demonstrates the concepts presented, and draws a comparison of ACF-invariance with other discretization schemes.

1. INTRODUCTION

Linear optimal filtering introduced by Wiener [8] is one of the most significant breakthroughs in the field of statistical signal processing. The work became the precursor to the later developments and advancements in the related areas [4]. Known forms of Wiener filter are either restricted to pure continuous-time (CT) domain or pure discrete time (DT) domain ([2], [5]). However, in many physical situations CT random signals are sampled and subsequently processed in DT [2]. In this paper we formulate the problem so as to preserve the optimal (Wiener) solution in this scenario. Accordingly, the result is called the hybrid Wiener filter.

The conventional approach to discretizing the Weiner design is to mimic the CT solution with sampled random processes. Known sampling schemes however, do not preserve the response of CT systems to random signals, which forms the basis of Wiener solution. In view of this, we develop an autocorrelation function (ACF) invariant DT equivalent of CT systems. This development subsequently admits the optimal causal IIR Wiener solution by spectral factorization.

An example illustrates the concepts. A comparison with mean square errors resulting from other commonly used discretization schemes highlights the significance of ACF-invariance.

2. THE HYBRID SCENARIO

Let $u_c(t)$ be a continuous random process. The process generator of $u_c(t)$ is assumed to be a linear time-invariant (LTI) causal stable filter $H(s)$ that is excited by zero-mean white noise $v(t)$ with variance σ_v^2. The autocorrelation function $r_{u_c}(t)$ of $u_c(t)$ is given by

$$r_{u_c}(t) = \sigma_v^2 \mathcal{Z}^{-1}\{H(z)H(1/z)\} \tag{1}$$

where $\mathcal{Z}^{-1}\{\cdot\}$ is the inverse Z-transform operator. The signal $u_c(t)$ is fed to an ideal sampler to obtain a discrete signal $u[n] = u_c(nT_s)$, where T_s is the sampling time. The process $u[n]$ is corrupted by added observation noise, $w[n]$, which is uncorrelated with $u[n]$, and sequence $s[n]$ is measured:

$$s[n] = u[n] + w[n] \tag{2}$$

Under the customary assumption that the observation noise is zero-mean, uncorrelated, $w[n]$ must be prescribed directly in DT because of the mathematical intractability of discretizing a CT white processes.

Let $d_c(t)$ indicate the desired signal (i.e. the signal to be estimated) ([3], [5], The process generator of $d_c(t)$ is again an LTI causal stable system $G(s)$, which is driven by zero mean white noise $v_1(t)$. We assume that $d_c(t)$ also undergoes ideal analog to digital conversion resulting in $d[n] = d_c(nT_s)$. We pass the signal $s[n]$ through a filter $H_{opt}(z)$, whose output is denoted by $y[n]$. The error sequence $e[n]$ is

$$e[n] = d[n] - y[n] \tag{3}$$

The hybrid Wiener filter $H_{opt}(z)$ is designed to minimize the mean squared error $E\left[e^2[n]\right]$, in which $E[\cdot]$ is the expectation operator Note that the output of $H_{opt}(z)$ is interpretable as an estimate of the desired signal and hence can be written as $y[n]=\hat{d}[n]$. The overall concept is illustrated in Figure 1.

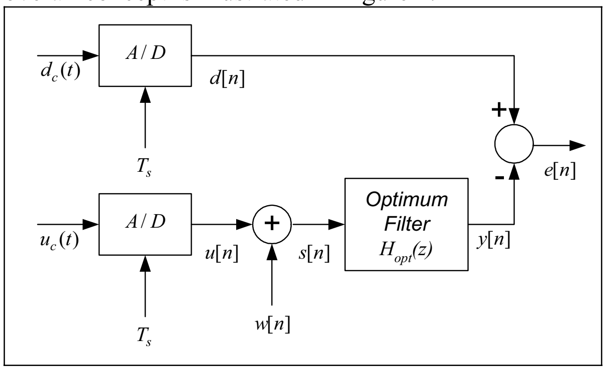

Figure 1. The hybrid Wiener filter problem

3. PROBLEM FORMULATION

The objective is to transform the hybrid filtering problem above into a purely DT one. In the case of a finite impulse response (FIR) Wiener filter, we get the normal equations by direct discretization of the ACF $r_{u_c}(t)$, yielding $r_u[k]=r_{u_c}(nT_s)$. A discussion of discretizing ACF follows in Section 4. We will not pursue the FIR Wiener filter because of its simple nature. The infinite impulse response (IIR) Wiener solution, however, requires knowledge of transfer function of the process generator (noise or signal?). This in turn necessitates the discretization of $H(s)$ which is the processor synthesizer of $u_c(t)$ in Figure 1. There are different methods for deriving finding DT equivalents of CT systems. Because of inherent loss of information in the discretization process, none of these schemes is universal. These methods are devised such that some frequency or time domain characteristic of the original continuous system is preserved in the DT domain. The commonly used methods are based on numerical integration, pole-zero mapping and/or hold equivalence. Examples are forward-backward difference, bilinear transformation, zero-order hold, first-order hold [2]. However, none of these methods takes into account the second-order statistical properties of the input and output signals. Accordingly, these DT equivalents are not suitable for the Wiener filtering problem, which is essentially a second-order optimal solution ([1], [8]).We propose a new scheme to find a discrete equivalent that preserves second-order statistical properties of a system. This concept is illustrated in Figure 2.

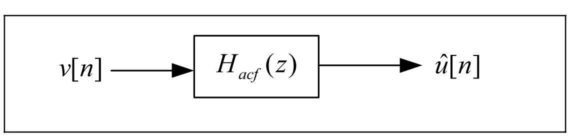

Figure 2. ACF-invariant discrete process generator

In the figure $v[n]$ is a discrete white sequence with variance σ_v^2 and $\hat{u}[n]$ is the output of the system $H_{acf}(z)$. Our goal is to find $H_{acf}(z)$ such that the ACF $r_{\hat{u}}[k]$ of the sequence $\hat{u}[n]$ has the property $r_{\hat{u}}[k]=r_{u_c}(nT_s)$. Accordingly we term $H_{acf}(z)$ as the ACF-invariant equivalent of $H(s)$. The power spectral density of $u[n]$ $(=\sigma_v^2 H_{acf}(z)H_{acf}(z^{-1}))$ and hence, that of $s[n]$, are now available. Consequently we can now find the noncausal Wiener solution or employ spectral factorization to arrive at the causal Wiener solution [5].

4. ACF-INVARIANT DISCRETE EQUIVALENT

Useful system functions in engineering tasks ordinarily proper rational. When the relative degree of $H(s)$ is zero, $u_c(t)$ in Figure 1 carries a component of white noise, which makes proper conversion to DT impossible. We therefore assume that *H(s)* is strictly proper. We also assume that $H(s)$ represents a real, causal, stable system. Let us begin with a second-order system with the following system function:

$$H(s)=\frac{k(s+z_1)}{(s+s_1)(s+s_2)} \tag{4}$$

The analysis also applies to a two-pole system [without the zero as in (4)],

$$H(s)=\frac{k}{(s+s_1)(s+s_2)} \tag{5}$$

For notational simplicity we drop the subscript on $r_{u_c}(t)$ and write $r(t)$. The conventional bracket notation is used to distinguish between CT, $r(t)$ and DT, $r[n]$. Using (1), it is found that ACF $r(t)$ for both systems (4) and (5) has the form

$$r(t)=\gamma\left[\alpha e^{-s_1|t|}+\beta e^{-s_2|t|}\right] \tag{6}$$

With T_s as the sampling time period, we discretize $r(t)$ and obtain $r[n]$ as follows

$$\begin{aligned} r[0]&=\gamma[\alpha+\beta] \\ r[\pm 1]&=\gamma\left[\alpha e^{-s_1T_s}+\beta e^{-s_2T_s}\right] \\ &\vdots \\ r[\pm k]&=\gamma\left[\alpha e^{-ks_1T_s}+\beta e^{-ks_2T_s}\right] \end{aligned} \tag{7}$$

It is easy to show that a general second order autoregressive AR(2) process is not consistent with (7), so that a DT "equivalent" of (5) would not produce an ACF-equivalent signal in DT. In an attempt to model (4), let us consider the autoregressive—moving-average ARMA(2,1) process, governed by the equation

$$u[n]+a_1u[n-1]+a_2u[n-2]=v[n]+b_1v[n-1] \tag{8}$$

The transfer function of the process is

$$H_{ARMA}(z)=\frac{\left(b_0+b_1z^{-1}\right)}{1+a_1z^{-1}+a_2z^{-1}} \quad (9)$$

In this case we have the following set of equations

$$\begin{aligned} r[0]+a_1r[-1]+a_2r[-2]&=\sigma_v^2\left(b_0h(0)+h(1)b_1\right)\\ r[1]+a_1r[0]+a_2r[-1]&=\sigma_v^2h(0)b_1 \quad (10)\\ r[m]+a_1r[m-1]+a_2r[m-2]&=0, \qquad m>1 \end{aligned}$$

where $h[n]$ is the impulse response of ARMA(2,1) process. We have

$$\begin{aligned} h[0]&=b_0\\ h[1]&=b_1-a_1b_0 \end{aligned} \quad (11)$$

From (7) and last equation of (10), equating coefficients gives the solution

$$\begin{aligned} a_1&=-\left(e^{-s_1T_s}+e^{-s_2T_s}\right)\\ a_2&=e^{-(s_1+s_2)T_s} \end{aligned} \quad (12)$$

An important observation is that a pole at $s=-p$ is mapped to a pole at

$$z=e^{-pT_s} \quad (13)$$

The same is observed for sampling schemes like hold equivalents and impulse invariance [2]. We are now in a position to solve the first two non-linear equations of (10). Dividing the two and rearranging yields

$$\left(\frac{b_1}{b_o}\right)^2-(\zeta+a_1)\left(\frac{b_1}{b_o}\right)+1=0 \quad (14)$$

where

$$\zeta=\frac{r[0]+a_1r[-1]+a_2r[-2]}{r[1]+a_1r[0]+a_2r[-1]}=\frac{r[0]+a_1r[1]+a_2r[2]}{(1+a_2)r[1]+a_1r[0]} \quad (15)$$

Solving (14) for b_1/b_o, we get two solutions. It can be shown that one is minimum phase and the other one is not. Finally b_0 and hence b_1 can be evaluated by equating DC gain of continuous system $H(s)$ and $H_{ARMA}(z)$ in (9). To ascertain the consistency of ARMA(2,1) with the data in (7), we have used the approach of matching coefficients. This method proves that the ARMA(2,1) process is consistent with (7) for all lags *k*. The solution of the normal equations ([3], [5]) for the ARMA process ($m=2,3$ in (10)), would have produced the same result but without a confirmation for $m>3$. After calculation of ARMA(2,1) parameters according to ACF-matching criterion, we use the notation $H_{acf}(z)$ for the resultant system function. Continuing the same approach, it can be shown that for complex conjugate poles with the system transfer functions

$$H(s)=\frac{k\left(s+z_1\right)}{\left(s+\zeta\right)^2+\phi^2}$$

or (16)

$$H(s)=\frac{k}{\left(s+\zeta\right)^2+\phi^2}$$

ARMA(2,1) process is sufficient to accurately meet the ACF matching condition. In this case the poles are mapped to

$$z=\kappa e^{\pm j\theta} \quad (17)$$

where $\kappa=e^{-\zeta T_s}$ and $\theta=\phi T_s$. Alternately

$$\begin{aligned} a_1&=-\kappa\left(e^{j\theta}+e^{-j\theta}\right)=-2\kappa\cos\left(\theta\right)\\ &=-2e^{-\zeta T_s}\cos\left(\phi T_s\right) \quad (18)\\ a_2&=\left|\kappa\right|^2=e^{-2\zeta T_s} \end{aligned}$$

The procedure for calculating b_0 and b_1 in (9) remains the same as in the case of real poles.

Generalizing the result of this discussion to a *p*-pole and *q*-zero (with $q<p$) continuous system $H(s)$, which is the process synthesizer for $u_c(t)$ in Figure 1, it can be shown that $u[n]$ is an ARMA$(p,p-1)$ process. In this case the autocorrelation sequence satisfies following set of equations [5]

$$\begin{aligned} r[m]+\sum_{l=1}^{p}a_lr[m-l]&=\sigma_v^2\sum_{l=0}^{p-1-m}h[l]b_{l+m}, \quad 0\le m\le p-1\\ r[m]+\sum_{l=1}^{p}a_lr[m-l]&=0, \qquad m\ge p \end{aligned}$$

(19)

The second set of equations for $m\ge p$ gives us results that maps a real pole at $s=-\kappa$ to a real pole at $z=e^{-\kappa T_s}$ and a complex conjugate pole-pair at $s=-\zeta\pm j\phi$ to a complex conjugate pole-pair at $z=e^{-\zeta T_s}e^{\pm j\phi T_s}$. The coefficients $b_l's$ must be calculated by solving the first *p* non-linear equations of (19) and equating the DC gains. The minimum phase solution is preferable so that $H_{acf}^{-1}(z)$ is causal and stable.

4.1. Noise Whitening Property

If $H_{acf}^{-1}(z)$ is causal and stable, then it acts like a noise-whitening filter in the DT domain. The concept is illustrated in Figure 3. Here $x[n]$ is a DT white sequence with variance $\sigma_x^2=\sigma_v^2$. This feature is not available for any other discrete equivalent. The noise-whitening property of the ACF discrete equivalent can prove to be valuable in identification of continuous systems when the observations are discrete.

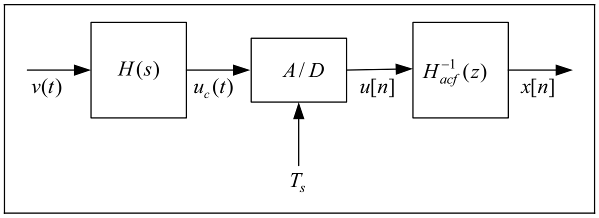

Figure 3. Noise-whitening property of $H_{acf}^{-1}(z)$

5. STRUCTURE OF HYBRID WIENER FILTER

The development of Section 4 enables us to transform the hybrid Wiener filter problem of Figure 1 into a pure discrete Wiener filter.

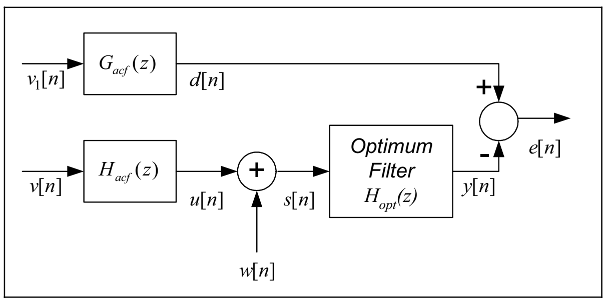

Figure 4. Discrete Equivalent of Hybrid Wiener Filter

In Figure 4, $H_{acf}(z)$ and $G_{acf}(z)$ are ACF invariant discrete equivalents of $H(s)$ and $G(s)$ respectively. Let the power spectral density $S_{ss}(z)$ of $s[n]$ be

$$S_{ss}(z) = \sigma_i^2 Q(z)Q(z^{-1}) \tag{20}$$

where $Q(z)$ is the minimum-phase part obtained by spectral factorization. The causal IIR Wiener filter is given by ([1], [5], [8])

$$H_{opt}(z) = \frac{1}{\sigma_i^2 Q(z)} \left[\frac{S_{ds}(z)}{Q(z^{-1})} \right]_+ \tag{21}$$

where $S_{ds}(z)$ is the cross power spectral density of $d[n]$ and $s[n]$. The subscript "+" indicates the causal part.

6. HYBRID WIENER FILTER EXAMPLE

Let $k = 1, s_1 = 1$ and $s_2 = 2$ in (5). Casting as an estimation problem we take $d(t) = u(t)$. With $T_s = 1\sec$, the method of Section 4 produces the following minimum phase ACF-equivalent DT system

$$H_{acf}(z) = \frac{0.45364(1+0.2049z^{-1})}{(1-0.5032z^{-1}+0.04979z^{-2})} \tag{22}$$

The following causal IIR optimal solution is obtained by spectral factorization

$$H_{opt}(z) = \frac{0.15666z(z+0.1104)}{(z^2-0.4071z+0.04199)} \tag{23}$$

For comparison, we discretize $H(s)$ by other commonly used methods. The mean square errors (MSE) of these cases are listed in Table 1. As expected ACF matching criterion gives the minimum MSE. The results in this example are not based on realizations of the random processes. This is in accordance with the fact that the Wiener solution requires actual ACF's and not the estimates.

Discretization Method	Mean Square Error
ACF Matching	0.1574
Zero Order Hold	0.1590
First Order Hold	0.1714
Bilinear (*fc=0.25Hz*)	0.2141
Impulse Invariance	0.1639
Pole-Zero Matching	0.1604

Table 1

7. CONCLUSION

An autocorrelation function (ACF) invariant DT equivalent signal model has led to the solution of the hybrid Wiener filter. This development is useful for DT processing of CT signals and systems. A discussion of time-domain (including transient behavior) and frequency-domain properties associated with ACF-invariance will appear in future work.

8. REFERENCES

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