# On Sampling a Subband of a Bandpass Signal by Periodically Nonuniform Sampling

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## ABSTRACT

Periodically nonuniform sampling (PNS) is employed to sample an interesting component over a subband in a bandlimited signal exclusive of using a physical filter. By controlling the sampling clock and the interpolation algorithm with respected to the interesting component, the sampling and frequency selection can be implemented simultaneously. The sampling rate and order are considered to merely allow the successful reconstruction of the interesting component and the average sampling rate is desirable to be lower than that to sample the entire signal. By analyzing the possible spectral aliasing cases in the selected band, it is found that the number of spectral replicas is at most 2 less than the maximum number in any subband of the signal. In addition, through the analysis, it is found that whatever the support of the selected band is, the average sampling rate has a lower bound. The limitation is primarily determined by the bandwidth of the complete signal.

## I. INTRODUCTION

It is well known that a continuous-time bandpass signal $x(t)$ can be reconstructed from its samples. Kohlenberg [1] originally showed that successful reconstruction from $x(nT_0)$, where $T_0 = 1/2B$ for bandwidth $B$ depends on the band position of $x(t)$ [2]. A necessary condition is that the positive lowest frequency $f_L$ (indicated in Fig. 1) must be an integer multiple of $B$. He also showed that the use of a particular form of sampling called periodically nonuniform sampling might remove these restrictions, and thus, $x(t)$ for any band position can be reconstructed from $x(nT_0)$. Consequently, periodically nonuniform sampling becomes an efficient way to reduce the sampling rate.

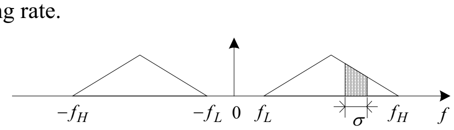


Fig.1. A bandpass signal. The component denoted by the dark area is the interesting component

In $X(f)$, if the component over a selected band for bandwidth $\sigma$ with $\sigma < B$, as shown in Fig.1, is of interest, generally, there are several ways to obtain the aliasing-free samples of the component: one way is that a band-pass filter is applied to $x(t)$ before sampling and correspondingly the sampling rate is determined only in terms of the bandwidth of the selected band. Obviously this way will have a less sampling rate at the cost of a hardware filter. The parameters of the filter have to be specified according to the location and range of the selected band. Y. Poberezhsky and G. Poberezhsky proposed a sampling technique to allowing exclusion of an anti-aliasing filter [3][4]. Antialiasing filtering can be performed by a multichannel sample-and-hold amplifier with weighted integration. This makes analogue-to-digital conversion much less expensive because multichannel SHAWIs are suited for IC technology. Besides, the antialiasing filtering by multichannel SHAWIS becomes adaptable to different bandposition and bandwidth by varying the weighing function as opposed to the conventional antialiasing filter. Another feasible way is that $x(t)$ is first sampled at a rate, which allows a perfect reconstruction of entire $x(t)$, and then the resultant samples are filtered using a digital bandpass filter. Thus, no other additional hardware except the necessary sampler is used in the overall system, and in this way we get the flexibility to obtain the interesting components. However, it acquires redundant samples and therefore requires higher sampling rate. If $x(t)$ is sampled at a rate that only allows the successful reconstruction of the interesting component, the sampling rate may be reduced. And furthermore, when the periodically nonuniform sampling is employed, only by controlling the sampling clock and the interpolation algorithm with respected to the interesting component, the sampling and frequency selection can be implemented simultaneously. Suppose $x(t)$ is periodically nonuniform sampled at the sampling rate of each channel $f_s$. The required minimum sampling order for perfect reconstruction of $x(t)$ is $N_B$ and the lower bound of the required minimum sampling order for the reconstruction of the interesting component is $N_\sigma$, $N_B$ and $N_\sigma$ for a given $f_s$ will be computed and analyzed in this paper. Furthermore, the average sampling rate for extracting the samples of the interesting component $f_\sigma = f_s \cdot N_\sigma$ will also be discussed in detail.

## II. RELATIONSHIP BETWEEN SAMPLING ORDER FOR SAMPLING THE ENTIRE SIGNAL AND SAMPLING RATE OF EACH CHANNEL

For periodically nonuniform sampling, each uniform sample stream is, in general, an undersampled representation of $x(t)$ and consequently is an aliased version of $x(t)$. Then, in the spectrum of each sample stream, a finite number of spectral replicas of $X(f)$ intersect with the in-band frequency of $X(f)$. To reconstruct $x(t)$, all unwanted replicas of $X(f)$ in the reconstructed band must be removed [5]. Coulson [6] has derived that only when the sampling order is larger than or equal to the number of replicas within the reconstructed band, it is possible to remove all the unwanted replicas and thus to reconstruct $x(t)$. In general, it is convenient to first define the subbands of $X(f)$, and then interpolates. Since not all the replicas intersecting with the reconstructed band may cover the whole band, i.e. the supports of some replicas are only on subbands in the reconstructed band, the numbers of replicas in subbands may be smaller than that of the replicas in the whole reconstructed band. Hence, the required sampling order will be less, and therefore the average sampling rate will be lower. The lower bound of the required sampling order that allows perfect reconstruction is called the minimum required sampling order, and it should be determined by the numbers of replicas within all the subbands. Generally, the lower or upper frequency boundaries of some replicas may be within the in-band of $X(f)$, and these boundaries naturally partition the whole band into several subbands. Within each of these subbands, the replicas intersecting on any frequency are the same, so any further division in these subbands is no helpful for the reduction of the required sampling order. Thus, the maximum number of replicas in these subbands is just the required minimum sampling order for perfect reconstruction of $x(t)$.

In principle, there exist two kinds of boundary frequencies within the reconstructed band: leftward boundary frequencies and rightward boundary frequencies. It is defined that if a boundary frequency is the lower or upper frequency limit of a spectral replica, it is a leftward or rightward boundary frequency. Since $X(f)$ consists of two disjoint conjugate symmetric bands: the positive component and the negative component, the leftward boundary frequencies and the rightward ones are both produced by the shifted replicas of these two bands. Hence, there are two kinds of leftward or rightward boundary frequencies. Suppose

$$\begin{cases} 2f_L/f_s = n+\alpha \\ B/f_s = m+\beta \end{cases} \quad (1)$$

where, $m$ and $n$ are integers and $0 \le \alpha, \beta < 1$, these boundary frequencies are defined as

$$\begin{cases} f_{pLk} = (\frac{n+\alpha}{2}+k)f_s, k=1,2,\Lambda, m+\lceil\beta\rceil-1 \\ f_{pRk} = (\frac{n+\alpha}{2}+\beta-\lceil\beta\rceil+k)f_s, k=1,2,\Lambda, m+\lceil\beta\rceil-1 \\ f_{nLk} = (\frac{n-\alpha}{2}-\beta+\lfloor\alpha+\beta\rfloor+k)f_s, \\ \qquad k=1,2,\Lambda, m+\lceil\alpha+2\beta\rceil-\lfloor\alpha+\beta\rfloor-1 \\ f_{nRk} = (\frac{n-\alpha}{2}+k)f_s, k=1,2,\Lambda, m+\lceil\alpha+\beta\rceil-1 \end{cases} \quad (2)$$

where, $f_{pLk}$, $f_{pRk}$ are the $k-th$ leftward and rightward boundary frequencies produced by the replicas of the positive component respectively, and $f_{nLk}$, $f_{nRk}$ are the $k-th$ leftward and rightward boundary frequencies produced by the replicas of the negative component respectively. Generally, for a subband, if its lower boundary is of the leftward kind, the replicas in it are one plus compared with those in its left adjacent subband, inversely, if its lower boundary is of the rightward kind, the replicas in it are one minus. Consequently, the number of spectral replicas in these subbands varies in a step of $\pm 1$. For a real bandpass signal, while undersampled, within the reconstructed band, boundaries of these two kinds are interleaved. Since the spectrum of the sampled signal of $x(t)$ is a periodic function of $f$ consisting of a superposition of shifted replicas of $X(f)$, scaled by $f_s$, each kind of boundary frequencies are uniformly spaced by an interval of $f_s$. Hence, there exist only three possible relative positions of these boundary frequencies, as shown in Fig.2. The frequencies indicated by "$\bullet$" are leftward boundary frequencies, and those indicated by "$\times$" are rightward boundary frequencies. Since $f_L$ is the lower frequency limit of $X(f)$, it is also considered as a leftward boundary frequency. "$+1$" and "$-1$" present one plus and one minus of spectral replicas respectively. It may be concluded from Fig.2 that within the frequency region between $f_L$ and the minimum one of the rightward boundary frequencies, the number of replicas is equal to $N_B$. That is

$$N_B = N\{f_L, \min(f_{nR1}, f_{pR1})\} \quad (3)$$

where, operator $N\{a,b\}$ denotes the number of spectral replicas of $X(f)$ within the frequency region $[a,b]$. It may be computed that, for arbitrary frequencies $a$ and $b$ in the in-band of $X(f)$,

$$\begin{aligned} N\{a,b\} = & \lceil (f_H-a)/f_s \rceil + \lceil (b-f_L)/f_s \rceil \\ & -\lfloor (f_L+a)/f_s \rfloor + \lceil (f_H+b)/f_s \rceil - 2 \end{aligned} \quad (4)$$

In addition, it can be obtained by computing (2) that

$$\min(f_{nRk}, f_{pRk}) = \begin{cases} f_{nR1}, \alpha+\beta \ge 1 \; or \; \beta=0 \\ f_{pR1}, \; otherwise \end{cases} \quad (5)$$

Thus, $N_B$ is found by substituting (4) and (5) into (3), that is

$$N_B = \begin{cases} N\{f_L, f_{nR1}\} = 2m, \beta=0 \\ N\{f_L, f_{nR1}\} = 2m+2\lceil\beta\rceil, \alpha+\beta \ge 1 \\ N\{f_L, f_{pR1}\} = 2m+\lceil\alpha+2\beta\rceil, otherwise \end{cases} \quad (6)$$

The average sampling rate may be expressed as

$$f_B = N_B.f_s = \begin{cases} 2B, \beta=0 \\ 2B+2(\lceil\beta\rceil-\beta)f_s, \alpha+\beta \ge 1 \\ 2B+(\lceil\alpha+2\beta\rceil-2\beta)f_s, otherwise \end{cases} \quad (7)$$

and we may find by analyzing (7) that there is a lower bound of $f_B$, that is

$$f_B \ge 2B \quad (8)$$

Only when $\beta=0$, i.e., the bandwidth $B$ are integer times of the sampling rate of each channel $f_s$, the average sampling is the lowest, $2B$, which is just the lower bound for any sampling scheme that allows perfect reconstruction [7].

Equations (6) and (7) imply that the minimum required sampling order and the minimum required average sampling rate for perfect reconstruction of $x(t)$ are closely related to the selection of sampling rate of each channel $f_s$ .

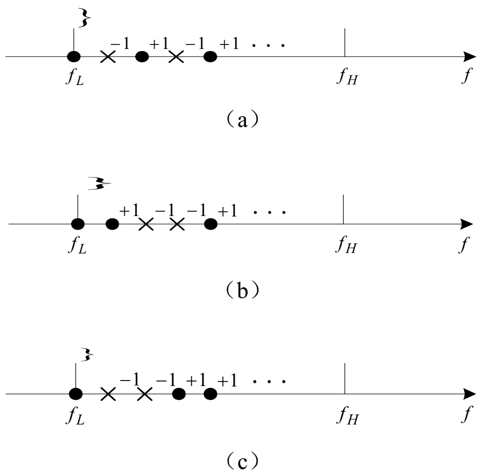


Fig.2. The boundary frequencies within the in-band of $X(f)$ .(a), (b) and (c) are three possible cases respectively

## III. RELATIONSHIP BETWEEN SAMPLING ORDER FOR SAMPLING THE ENTIRE SIGNAL AND THE SELECTED BAND

$N_B$ is the required minimum sampling order for perfect reconstruction of $x(t)$ , it means that any component over the in-band of $X(f)$ can be recovered if the sampling order is equal to $N_B$ or higher. The interesting component is just over a band within the in-band of $X(f)$ , consequently, $N_\sigma$ is not larger than $N_B$ . On the other hand, from Fig.2, it may be found that, if $N_B > 2$ , in any subband, the replicas are at most 2 less than the maximum number of replicas within all the subbands, i.e., $N_B$ , so $N_\sigma$ can not be less than $N_B - 2$ . Thus,

$$N_B \geq N_\sigma \geq N_B - 2 \tag{9}$$

Since the interesting component is just over a band within the in-band of $X(f)$, we may obtain $N_\sigma$ by analyzing the minimum sampling order that allows the reconstruction of a subband within the in-band of $X(f)$ . Obviously, the minimum sampling order which allows the reconstruction of a subband is also the lower bound of the sampling order for reconstruction of the interesting component, i.e., $N_\sigma$ .

Based on above analyses about the boundary frequencies within the in-band of $X(f)$ , $N_\sigma$ may be described under three possible cases: 1) if $\beta = 0$ , where $\beta$ is defined by (1), the rightward boundary frequencies and the leftward ones match together, and therefore the numbers of replicas within each subband are equal, so they are all equal to $N_B$ . It means that any component of $X(f)$ can be recovered only when the sampling order is not less than $N_B$ . Hence, $N_\sigma = N_B$ ; 2) when $\alpha + \beta \leq 1$ and $\alpha + 2\beta > 1$ , where $\alpha$ and $\beta$ are defined by (1), two kinds of rightward boundary frequencies are adjacent to each other (as shown in Fig.2(b),(c)). Here, $N_B > 2$ and there will exist at least one subband within which the number of replicas is equal to $N_B - 2$ . Then, if the selected band is within such subband, the required sampling order is the lowest, i.e., $N_\sigma = N_B - 2$ ; 3) for $\alpha$ and $\beta$ the rest values, the rightward boundary frequencies and the leftward ones interleave (as shown in Fig.2 (a)), and therefore the minimum number of replicas within the subbands is $N_B - 1$ . Similarly, here, we may get that $N_B > 1$ and $N_\sigma = N_B - 1$ . Thus, the relationship between $N_\sigma$ and $N_B$ can be expressed as

$$N_\sigma = \begin{cases} N_B, \beta = 0 \\ N_B - 2, \alpha + \beta \leq 1 \ and \ \alpha + 2\beta > 1 \\ N_B - 1, otherwise \end{cases} \tag{10}$$

Appropriate selection of $f_s$ can make the selected band be contained by a subband within which, the number of replicas is minimum, and thus, $N_\sigma$ defined by (10). is obtained.

## IV. ANALYSIS ON THE MINIMUM REQUIRED SAMPLING RATE FOR SAMPLING THE SELECTED BAND

Substitution of (6) into (10) produces that

$$N_\sigma = \begin{cases} 2m, \beta = 0 \\ 2m + \lceil \alpha + 2\beta \rceil - 1, \alpha + \beta \leq 1 \\ 2m + 2\lceil \beta \rceil - 1, \alpha + \beta > 1 \end{cases} \tag{11}$$

and then

$$f_\sigma = f_s \cdot N_\sigma = \begin{cases} 2B, \beta = 0 \\ 2B - (1 + 2\beta - \lceil \alpha + 2\beta \rceil) f_s, \alpha + \beta \leq 1 \\ 2B - (1 - 2\beta) f_s, \alpha + \beta > 1 \end{cases} \tag{12}$$

Equation (12) shows that $f_\sigma$ is still primarily determined by the frequency location and bandwidth of the complete signal $x(t)$ , together with the sampling rate of each channel $f_s$ .

We can get the bound of $f_\sigma$ from (12) that

$$2B \geq f_\sigma \geq 2B - f_s \tag{13}$$

while $\alpha = 0$ and $\beta = 1/2$ , $f_\sigma$ is equal to $2B - f_s$ .

During the discussion in section III, the following relational expression may also be obtained

$$\begin{cases} N_\sigma = N_B, N_B \geq 1 \\ N_\sigma = N_B - 1, N_B \geq 2 \\ N_\sigma = N_B - 1, N_B \geq 3 \end{cases} \tag{14}$$

Comparison of (14) with (6) and (10) produced that the necessary condition to satisfy (14) is

$$m \geq 1 \tag{15}$$

and therefore

$$B \geq f_s \tag{16}$$

Thus, inequality (13) is changed to

$$2B \geq f_\sigma > B \tag{17}$$

(17) indicates that $f_\sigma$ will always be smaller than $f_B$ except that when $\beta = 0$ , and it also shows us，whatever the support of the selected band is, $f_\sigma$ should be larger than the bandwidth of the complete signal. Consequently, it may be concluded that the scheme of sampling the signal at a sampling rate that only allows the reconstruction of the interesting

component really can reduce the required sampling rate, whereas there is a lower bound of the sampling rate and the bound will not be determined by the support of the selected band, but the bandwidth of the entire signal.

In section II, we have obtained that when $\beta = 0$, the required average sampling rate $f_B$ is minimum, therefore, while sampling the entire signal $x(t)$, $f_s$ is usually chosen to make $B$ integer times of $f_s$, so as to reduce the sampling rate. However, in this section, it can be found that when $\beta = 0$, $f_\sigma = 2B$, which is just the maximum one. Consequently, that $B$ are integer times of $f_s$ is disadvantageous to sampling an interesting component in $X(f)$.

## V. DISCUSSION

According to above analyses, the required average sampling rate to obtain the aliasing-free samples of the interesting component $f_\sigma$ can be much lower than that to obtain the aliasing-free samples of the entire signal $f_B$. Although $f_\sigma$ is not primarily determined by the bandwidth and band location of the selected subband, the selection of the optimal sampling parameters (the sampling rate of each channel and the sampling order), which will produce the minimum average sampling rate, may be closely related to them by following reasons: while $\alpha$ and $\beta$ that represent the band position and bandwidth respectively of $X(f)$ relative to the sampling rate of each channel $f_s$ satisfy certain condition, there possibly exists the corresponding minimum required average sampling rate $f_\sigma$ described by (12); however, such $f_\sigma$ can be gotten only when the selected band is within one of the special subbands, within which, the numbers of spectral replicas are minimum. The position and bandwidth of these special subbands depend on $\alpha$ and $\beta$, while $\alpha$ and $\beta$ is uniquely determined by the sampling rate of each channel $f_s$ for the given $X(f)$. That is, $f_s$ determines the position and range of these special subbands. To obtain the minimum sampling rate, $f_s$ should be chosen to make the special subbands contain the selected band. Thus, the sampling parameters are associated with the selected band position and range along with the frequency support of the entire signal. Based on this, the optimal sampling parameters should be chosen under following rules: firstly, the selected band must be contained in the special subbands within which the numbers of replicas are the minimum. This confines $\alpha$ and $\beta$ into some special ranges and resultantly confines $f_s$ into some special range. Secondly, chose one within the required range of $f_s$ that make corresponding $f_\sigma$ defined by (12) the minimum.

Since for periodically nonuniform sampling, the sampling clock determines the sampling parameters; the sampling and frequency selection can be implemented simultaneously by controlling the sampling clock and the interpolation algorithm with respected to the interesting component. Thus, there is no additional hardware except the necessary sampler in the overall system, which makes the system more flexible, even more flexible and .more costless than the system proposed in [3].

On the other hand, the lower bound of $f_\sigma$ is the bandwidth of the entire signal $x(t)$, as derived in section IV, when $x(t)$ is a broadband signal, this frequency-selective sampling scheme is of less advantage, for that the required average sampling rate will be very high. Consequently, although the proposed frequency-selective sampling scheme gets the flexibility to obtain the interesting components, there exist some critical conditions under which this scheme can be employed

## VI. CONCLUSION

An approach of sampling a selected subband in a bandpass signal exclusive of using a hardware filter before sampling can provide great flexibility and adaptation in digitizing application. However, the cost is that the higher sampling rate is required. Periodically nonuniform sampling is used in such approach, so as to reduce the required sampling rate, since this sampling scheme let any bandpass signal be recovered from its samples at a sampling rate of $2B$. It is concluded from above analyses that compared with the sampling rate to sample the entire signal, the sampling rate to sample an interesting component of the entire signal may be lower. However, for the most part the sampling parameters are not uniquely determined by the selected subband position and the range, it also closely related to the specification of the complete signal, therefore, there is a lower bound of $f_\sigma$, that is, the bandwidth of the entire signal. It is indeed lower than the minimum sampling rate to sample the entire signal. But it clearly states that sampling an interesting subband in a signal by PNS exclusive of an antialiasing filter, in some cases, the average sampling rate is not economic compared with sampling the entire signal.

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