A METHOD FOR EFFICIENT INTERPOLATION OF DISCRETE-TIME SIGNALS BY USING A BLUE-NOISE MAPPING METHOD

Zeljko Ignjatovic, Mark F. Bocko

University of Rochester, Rochester NY, USA

ABSTRACT

A method for the efficient interpolation of uniformly sampled discrete time signals is described. The method can be used to interpolate N-dimensional discrete signals such as audio, images, and 3D medical imaging. As opposed to conventional interpolation methods, where uniform zero-insertion (expansion) is performed prior to low-pass filtering, the proposed method inserts zeros in a blue-noise pattern, which has been shown most appropriate for this application. This interpolation method attenuates undesired replicas solely by the act of noisy zero-insertion, resulting in relaxed requirements on the order of the interpolation filter, thereby reducing the required computations. This work shows that an N^{th} order blue noise pattern reduces the filter order by N.

1. INTRODUCTION

The interpolation of digital signals, with various interpolation filters in terms of order and linearity, is a well-established procedure, [1,2]. The procedure has been used for decades to prepare input signals for digital-to-analog conversion based on the sigma-delta (ΣΔ) modulation method, to increase an image's spatial resolution so that it is suitable for printing, and to estimate continuous-tone color values of an image during image acquisition (i.e., in CFA – Bayer pattern based digital cameras, an image is a mosaic of 3 colors; thus image acquisition is followed by interpolation to "demosaic" the image). There are many proposed interpolation techniques, such as Inverse Distance Weighting, Trend Surface, Splines, and Kriging. Along with conventional interpolation of uniformly sampled discrete time signals, the interpolation of stochastically sampled signals has been developed as well, with methods such as the Voronoi-Alebach interpolation technique [3,4]. Furthermore, it has been proven useful to sample certain physical phenomena stochastically rather than uniformly. For a stochastic sampling pattern with blue-noise characteristics, it has been shown that the aliasing artifacts are scattered throughout the spectrum out of the signal band, and that they appear as broad-band noise that is easily filtered out, [5].

In this work, we introduce a stochastic approach for the interpolation of uniformly sampled discrete signals. This blue-noise interpolation (BNI) technique scatters undesired replicas throughout the spectrum, reducing the demands on the interpolation filter. We will observe the application of the proposed algorithm for 1D signals. However, the proposed interpolation procedure may easily be extended to N-dimensional signals.

2. OVERVIEW OF CONVENTIONAL INTERPOLATION ALGORITHMS

Interpolation is defined as a two-step procedure, where an input digital signal is first L-fold expanded and then low-pass filtered, as shown in Fig. 1. The resulting interpolated digital signal z[n] is an L-times upsamled version of the input digital signal x[n]. In the ideal case, where an ideal low-pass filter is used, the resulting digital signal z[n] is equivalent to the signal obtained by sampling the continuous analog input signal x(t) at a rate that is L times higher than the sampling rate of the digital input signal x[n] that undergoes interpolation.

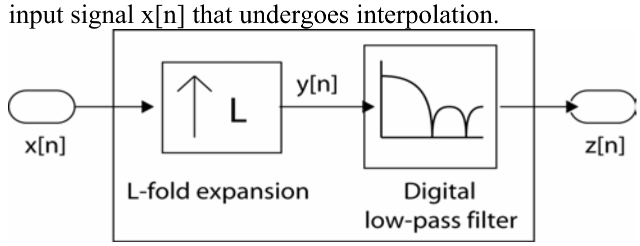

Figure 1: L-fold interpolation procedure

The first step of the conventional interpolation procedure is an L-fold expansion, or upsampling, defined by Eq. (1).

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right], & \text{if } n/L \text{ is an integer,} \\ 0, & \text{if } n/L \text{ is noninteger,} \end{cases} \qquad (1)$$

The L-fold expansion can also be described as a zero-insertion procedure, where L-1 zeros are uniformly inserted between the L samples of the input signal x[n]. The expansion operation is linear; however, it is not time-invariant. The Fourier transform of the expanded sequence Y(jθ) can be expressed in terms of X(jθ), as in Eq. (2).

$$Y(j\theta) = X(jL\theta) \qquad (2)$$

Figure 2 illustrates a typical spectrum of the expanded sequence Y(jθ), which is a 4-fold expansion of the input sequence X(jθ). The expansion procedure creates undesired replicas of the input signal spectrum. The number of replicas is equal to the number of inserted zeros per sample. Also, the replicas are highly correlated with the baseband signal (i.e., they have the same spectral shape and power). In practice, a low-pass filter that filters out the undesired replicas almost always follows the expansion procedure. However, the required low-pass filtering places a computational burden on the interpolation procedure, which is exacerbated in audio applications, where heavy filtering is usually required prior to ΣΔ digital-to-analog conversion (DAC), [6].

3. BINARY BLUE-NOISE PATTERNS AND BLUE NOISE INTERPOLATION

In typical audio and image signals the power of the input signal is usually concentrated in the lowest few frequency bins creating strong spikes in the input signal spectrum. The undesired replicas created by the process of expansion are also strong spikes by themselves. In order to remove these spikes heavy filtering is usually required. However, this usually is not acceptable due to its computational cost (i.e. only lower order filters are allowed). Thus, the remaining spikes might cause blocking effects in imaging applications or an overload of the ΣΔ DAC in audio applications. Thus, the filtering procedure remains the most computationally intensive step during conventional interpolation with the order of the filter being a tradeoff between the required attenuation of the replicas and the computational costs associated with filtering.

The L-fold expansion procedure may be described as a mapping in which the input signal samples are mapped to a binary pattern of "ones" and "zeros" with mean value equal to 1/L, such that the signal samples are mapped to the "ones". In the conventional expansion procedure, the binary mapping pattern is periodic, which eventually causes undesired replicas.

In order to reduce undesired replicas, we propose a randomization of the periodic binary mapping pattern in which the "ones" that correspond to the input signal sample locations are randomly interchanged with neighboring zeros. When the input signal samples are mapped to the randomized binary pattern, the undesired replicas are spread and de-correlated from the baseband signal.

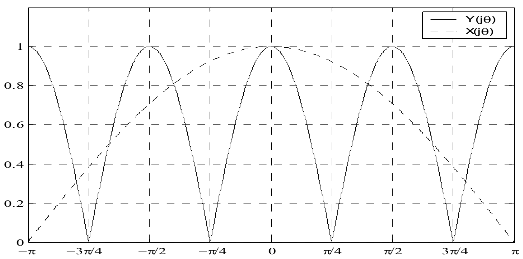

Figure 2: Spectrum of an input signal and its 4-fold expansion

The resulting binary mapping pattern has a blue-noise spectral shape. A blue-noise (BN) pattern is a stochastic model for describing ideal aperiodic patterns, where the spectrum of such a pattern displays no coherent spikes and has a deficiency of low-frequency energy.

We propose a blue-noise interpolation (BNI) procedure in which the input signal samples are mapped to a binary BN pattern instead of a periodic pattern. BNI improves the expansion step in comparison to conventional interpolation by spreading undesired replicas into a blue-noise (out of the baseband signal) with no coherent strong spikes. As a result, the requirements on the interpolation filter order are relaxed.

BNI also may be described as random zero-insertion, as opposed to periodic zero-insertion in the conventional procedure. In order not to corrupt a baseband replica, the zero-insertion noise pattern has to have a deficiency of low-frequency energy, implying that the most suitable zero-insertion pattern has BN spectral characteristics. Therefore, the baseband signal is not changed and the undesired replicas appear as high-frequency random noise rather than highly correlated replicas of the baseband signal. The statistical properties of the BN mapping pattern determine the spectral characteristics of the resulting broadband noise.

4. BLUE NOISE OBTAINED BY JITTERING

The most common method to create a BN binary pattern is to apply jitter to a binary periodic pattern, [5]. To demonstrate the benefits of BNI, we examine a 2-fold BNI process, where the binary BN sequence e[n] is obtained by applying jitter to the periodic signal $e_p[n]$, given by Eq. (3). Thus, a BN sequence is generated such that each "one" in the periodic sequence undergoes an equal probability shift in position of either 0 or +1. As a result, the "ones" remain either at their original position or they interchange with following "zeros".

$$e_p[n] = \frac{1 + (-1)^n}{2} \qquad (3)$$

The autocorrelation function and the power spectral density of the BN sequence e[n] are given by Eq. (4) and (5), respectively.

$$R_e[n] = \frac{1}{4}\left(1 + \delta[n] - \frac{1}{2}\left(\delta[n-1] + \delta[n+1]\right)\right) \quad (4)$$

$$S_e(\theta) = \frac{\pi}{2}\delta(\theta) + \frac{1}{4}(1 - \cos\theta) \quad (5)$$

In the following, it is proven that when the input signal x[n] is 2-fold expanded by the BN sequence e[n], the spectrum of the resulting signal $y_{BN}[n]$ has reduced amplitude replicas as compared to the conventional 2-fold expanded sequence. Furthermore, it is shown that the replicas are attenuated the same as in the conventional 2-fold expansion followed by the sample-and-hold filter. The input signal x[n] and the 2-fold BN expanded signal $y_{BN}[n]$ are related as shown in Eq. (6).

$$x[n] = y_{BN}\left[2n + \Delta(n)\right] \quad (6)$$

where Δ[n] is the shift at the sample point 2n. The shift Δ is a random variable with a uniform probability density function ($p_\Delta(k)=0.5$ for k=0,1). Thus, the spectrum of the BN expanded signal $y_{BN}[n]$ is given by Eq. (7).

$$Y_{BN}(j\theta) = \sum_{n=-\infty}^{\infty} y_{BN}[n]e^{-jn\theta} = \sum_{n=-\infty}^{\infty} y_{BN}\left[2n + \Delta(n)\right]e^{-j(2n+\Delta(n))\theta}$$
$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j2n\theta}e^{-j\Delta(n)\theta} \quad (7)$$

The expected value of the power spectral density of the BN expanded signal $y_{BN}[n]$ is given by Eq. (8).

$$E\left[\left|Y_{BN}(j\theta)\right|^2\right] = \left|X(j2\theta)\right|^2 E\left[e^{-j\Delta(n)\theta}e^{j\Delta(m)\theta}\right]$$
$$= \left|X(j2\theta)\right|^2 \frac{1+\cos\theta}{2} \quad (8)$$

Let the sequence $y_{C\_SH}[n]$ be the 2-fold conventionally expanded x[n] followed by the sample-and-hold filter (zero-order interpolation). The power spectral density of the sequence $y_{C\_SH}[n]$ is given by Eq. (9).

$$\left|Y_{C\_SH}(j\theta)\right|^2 = \left|X(j2\theta)\right|^2 \cdot 4\cos^2\left(\frac{\theta}{2}\right)$$
$$= \left|X(j2\theta)\right|^2 4\frac{1+\cos\theta}{2} \quad (9)$$

We conclude that the BN expansion, without any filtering, attenuates undesired replicas as a raised-cosine low-pass filter (Eq. (8)), which is the same as the conventional 2-fold expansion followed by a sample-and-hold filter, Eq. (9).

Figure 3 shows an example of the spectrum of a 64-sample cosine expanded 2-fold by conventional interpolation and through the BN interpolation method. According to Eq. (8), a 26dB attenuation of the replica at the cosine frequency should exist. In this example, a 15dB attenuation was achieved, which is less than predicted and is due to the short length of the signal (i.e., there is significant spectrum leakage). However, we have shown that the predicted 26dB attenuation is achieved in the expansion of a 256-sample (and longer) sequences. Also, we have observed that BN expansion does not degrade the phase of the baseband signal.

We may extend the BNI procedure by comparing the ensemble of the BN expanded sequences followed by the sample-and-hold and the conventional first-order interpolation (bilinear interpolation). The BN expanded sequence followed by the sample-and-hold filter (y_{BN_SH}) can be presented in terms of the conventional 2-fold expanded sequence $y_C[n]$ ($x[n]= y_C[2n]$), Eq. (10).

$$y_{BN\_SH}[2n] = y_C\left[2n - 2\Delta(n)\right],$$
$$y_{BN\_SH}[2n+1] = y_C[2n] \quad (10)$$

The spectrum and expected value of the power spectral density of $y_{BN\_SH}[n]$ are given by Eq. (11) and (12), respectively.

$$Y_{BN\_SH}(j\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2n\theta}\left(e^{-j2\Delta(n)\theta} + e^{-j\theta}\right) \quad (11)$$

$$E\left[\left|Y_{BN\_SH}(j\theta)\right|^2\right] = \left|X(j2\theta)\right|^2(1+\cos\theta)^2 \quad (12)$$

Let the sequence $y_{C\_BL}[n]$ be the 2-fold conventionally expanded x[n] followed by the bilinear filter. The spectrum and power spectral density of the signal $y_{C\_BL}[n]$ are given by Eq. (13) and (14), respectively.

$$Y_{C\_BL}(j\theta) = X(j2\theta)(1+\cos\theta) \quad (13)$$

$$\left|Y_{C\_SH}(j\theta)\right|^2 = \left|X(j2\theta)\right|^2(1+\cos\theta)^2 \quad (14)$$

We conclude that the BN expansion, followed by the sample-and-hold, attenuates undesired replicas as a raised-cosine-squared low-pass filter, which is the same as the conventional 2-fold expansion, followed by the bilinear filter, Eqs. (12) and (14).

In light of the above analysis, we conclude that by performing the BN zero-insertion process, obtained through uniform jittering, the filter requirement is relaxed by one order.

We have seen that the resulting attenuation of the high-frequency replicas during zero-insertion is defined by the spectral properties of the BN sequence (i.e. BN sequence defined by Eq. (5) acts as a raised-cosine). It is expected that if the BN sequence is better, in the sense of the sharpness of the spectrum cutoff, it would potentially attenuate replicas even more, thus further reducing the requirements placed on the interpolation filter. An example of the BN sequence that has sharper roll-off than the BN sequence generated by uniform jitter is that generated by a 2^{nd}-order ΣΔ modulator structure. As shown in [7], the output of a one-bit 2^{nd}-order ΣΔ modulator with the input signal equal to zero is quantization noise shaped by the double differentiation function, the spectrum of which is given by Eq. (15), the sum of a delta function and a broadband component.

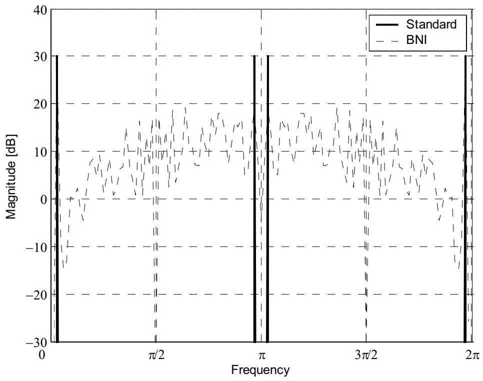

Figure 3: Spectrum of 64-sample cosine 2-fold expanded by conventional and BN expansion method

$$S_e(\theta)=\frac{\pi}{2}\delta(\theta)+N_0\left(6-8\cos\theta+2\cos 2\theta\right) \quad (15)$$

Figure 4 compares the broadband parts of the BN sequences generated by "jittering" and by 2nd-order ΣΔ modulation. The latter BN pattern has lower low and middle frequencies in the broadband component. Thus, we infer that it has the potential to push zero-insertion noise even further from the baseband spectrum.

Simulations show, see Fig. 5, that the 2nd-order BN zero-insertion procedure attenuates undesired replicas by an amount equivalent to bilinear interpolation, thus reducing the interpolation filter order by 2. Thus, our conjecture is that the BN sequence generated by an N^{th}-order ΣΔ modulator may attenuate replicas the same as an N^{th}-order filter, thus reducing the interpolation filter order by N.

The aforementioned analyses hold for any L-fold expansion. In such a case, the BN sequence is generated such that Δ[n] takes value 0, 1,.., L-1 with probability 1/L.

5. CONCLUSION

A new interpolation method based on a blue noise zero-insertion technique has been proposed and analyzed. Mathematical analyses show an improvement in the attenuation of undesired replicas of the baseband signal arising from zero-insertion while performing interpolation. It has been shown that the improvement depends strongly on the sharpness of the low frequency roll-off in the spectrum of the blue noise sequence used in the zero-insertion procedure. If the blue noise sequence is generated by the uniform jittering procedure, the interpolation filter order is relaxed by one order and for blue noise sequences generated by an N^{th} -order ΣΔ modulator the filter order requirement is apparently relaxed by N orders. In an ongoing effort, we are seeking to demonstrate the potential advantages of BNI for multi-dimensional applications, such as images.

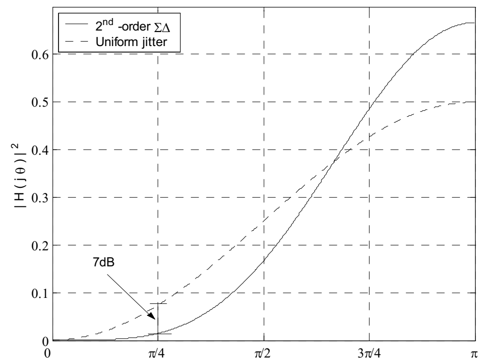

Figure 4: Power spectral density of different BN sequences

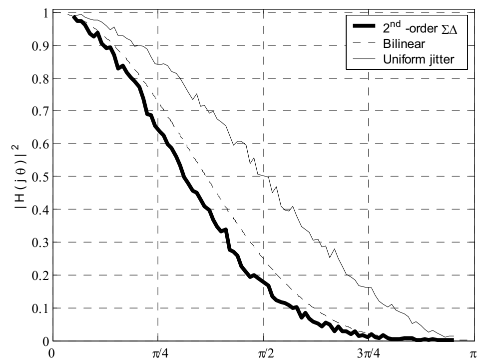

Figure 5: Replica attenuation properties of BNI (uniform jitter and 2nd-order ΣΔ) and conventional bilinear interpolation

6. REFERENCES

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