OPTIMAL FILTER BANK RECONSTRUCTION OF PERIODICALLY UNDERSAMPLED SIGNALS

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ABSTRACT

A multirate filter bank model is considered for reconstruction of periodically sampled signals. In contrast to many previous methods which considered perfect reconstruction of deterministic signals, this paper's approach uses a known discrete-time cyclostationary signal model to find a minimum mean-squared error reconstruction solution. A primary advantage of this approach is that it does not require a minimum sampling density, allowing optimal solutions to be determined in cases of undersampling. This allows for consideration of a wide range of generalized sampling problems under a single framework. An example with simulation results is presented.

1. INTRODUCTION

Since the introduction of Papoulis's generalized sampling expansion (GSE) formula [1], there has been much work examining reconstruction solutions to periodic sampling problems. Of particular interest are techniques making use of multirate filter banks. Established results in perfect reconstruction (PR) filter bank theory [2] can be viewed as a discrete-time version of generalized sampling. In the case of bandlimited signals with sufficiently dense sampling, the discrete-time filter bank methods are equivalent to standard continuous-time generalized sampling. However, the discrete-time filter bank approach has a distinct advantage in terms of feasibility, since a digital implementation is more readily designed.

Some previous filter bank solutions to various periodic sampling problems include [3–9]. These approaches consider deterministic signals with known finite spectral support. Periodic sampling at a sufficient average density allows a PR solution to be determined. These solutions seek to determine reconstruction filter designs that eliminate aliasing effects caused by nonuniform sampling, or in the most general case, aliasing and distortions resulting from generalized sampling.

This paper considers periodic sampling of a stationary random process with a known power spectral density (PSD). Instead of finding a PR solution, a reconstruction solution minimizing the time-average mean-squared error (TAMSE) is determined. Because this alternate criterion is used, the bandwidth limitations necessary for a PR solution are not required. While this approach can be used to determine a PR solution if sufficiently sampled bandlimited signals are considered, the focus of this paper will be on the case of periodically nonuniform undersampling.

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This paper is organized as follows. Section 2 presents a discrete-time model for the sampled signal and the periodic sampling process. The optimal reconstruction solution for this model is presented in Section 3 and an example is discussed in Section 4.

2. MODEL SETUP

The sampling model uses a discrete-time input signal $x[n]$ to represent a continuous-time random process $x(t)$, sampled uniformly with spacing T_M. This model assumes $x(t)$ is bandlimited to $\beta = \pi/T_M$. Since the spacing T_M is used only to define a high-resolution model and does not represent a physical sampling process, it can be made arbitrarily small such that $x[n]$ accurately represents a real-world signal. The PSD of $x[n]$ is determined from that of $x(t)$.

The signal $x[n]$ is applied as the input of the filter bank depicted in Fig. 1. Each of the C subbands represents an individual uniform sampler operating at T_P. This sampling period is related to the model sampling period T_M through $T_P = DT_M$, where D is the decimation factor of the filter bank. Typically T_M will be selected such that $D > C$. The analysis bank is used to model a physical periodic sampling process of C samples per T_P. Under a generalized sampling approach, any set of linear time-invariant analysis filters can be considered. However, this paper will consider a pure sampling process, for which the individual analysis filters are simply delays. The frequency response of the kth analysis filter is $H_k(e^{j\omega}) = e^{-j\omega d_k}$, meaning the analysis portion of the kth subband models a uniform sampler collecting $x(nDT_M - d_kT_M) = x(nT_P - d_kT_M)$. The combination of these C uniform sampling processes forms a periodic sampling process.

In this paper the model is assumed free from noise and quantization effects. A minimum MSE solution in the presence of quantization noise was developed in [10] under a bandlimited GSE framework. Noise analysis using this paper's framework will be considered in [11].

The output signal $y[n]$ is formed by passing the periodic samples of $x[n]$ through the synthesis bank. Since a multirate filter bank will generally form a linear periodically time-varying system, $y[n]$ must be considered to be a cyclostationary process with period D [12]. A cyclostationary framework from [13] will be used to determine optimal reconstruction filters (synthesis filters). This framework allows for consideration of both $x[n]$ and $y[n]$ as discrete-time wide-sense cyclostationary signals with period D, or WSCS(D). Such signals have periodically stationary first- and

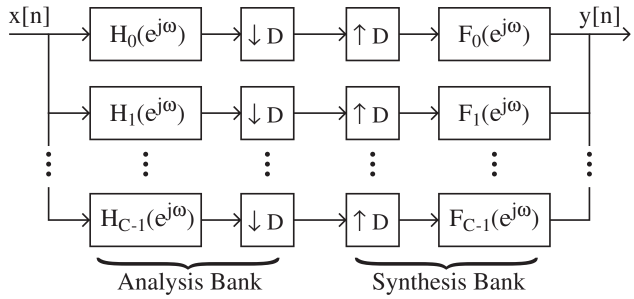

Fig. 1. Multirate filter bank structure used for periodic sampling and reconstruction.

second-order expectations. That is,

$$\begin{aligned} E\left[x[n]\right] &= E\left[x[n+kD]\right] \\ R_{xx}[n,m] &= R_{xx}[n+kD, m+kD], \end{aligned} \tag{1}$$

for every integer k, where $R_{xx}[n,m] = E\left[x[n]x^*[m]\right]$. Herein, this paper assumes the WSCS(D) signals have zero mean. While in many cases a wide sense stationary (WSS) input will be considered, a cyclostationary model is still required due to the fact that the output is not necessarily WSS. Since WSS processes are WSCS(D), this also serves for generalization purposes.

3. OPTIMAL RECONSTRUCTION SOLUTION

A set of reconstruction filters minimizing the time-averaged variance of $e[n] = x[n] - y[n]$ (i.e., the TAMSE) is sought. This variance is defined as

$$\sigma_e^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} E\left[e[n]e^*[n]\right] = \frac{1}{D}\sum_{n=0}^{D-1} R_{ee}[n,n]. \tag{2}$$

A relationship between the cyclostationary statistics of $x[n]$ and $y[n]$ is found in [13]. This relationship will be used to determine synthesis filters minimizing (2).

The cyclic correlation function of $x[n]$ is found through

$$R_{xx}^{\alpha}[u] = \frac{1}{D}\sum_{k=0}^{D-1} R_{xx}[k+u,k]e^{-j2\pi\alpha k}, \tag{3}$$

for $\alpha = n/D$ and integer n. The discrete-time Fourier transforms of these cyclic correlations produce the cyclic spectral densities (CSDs) through

$$S_{xx}^{\alpha}(e^{j\omega}) = \sum_{u=-\infty}^{\infty} R_{xx}^{\alpha}[u]e^{-j\omega u}. \tag{4}$$

The CSD input/output relationship of a filter bank is found using a matrix representation for the functions defined in (4). The (p,q)th element of the $D\times D$ CSD matrix is defined by

$$\left[\mathbf{S}_{xx}(e^{j\omega})\right]_{(p,q)} = S_{xx}^{(p-q)/D}(e^{j\omega}W^p) \quad \text{over } |\omega| \le \pi/D, \tag{5}$$

for $p,q = 0,\cdots,D-1$ where $W = e^{-j(2\pi/D)}$. This representation is only valid for $|\omega|\le\pi/D$ and completely represents the D distinct CSDs defined by (4).

The analysis and synthesis filters are represented using alias component (AC) matrices [2]. The AC matrix representations are $\mathbf{H}_{AC}(e^{j\omega}) =$

$$\begin{bmatrix} H_0(e^{j\omega}) & \cdots & H_{C-1}(e^{j\omega}) \\ \vdots & \ddots & \vdots \\ H_0(e^{j\omega}W^{(D-1)}) & \cdots & H_{C-1}(e^{j\omega}W^{(D-1)}) \end{bmatrix} \tag{6}$$

for the analysis bank and $\mathbf{F}_{AC}(e^{j\omega}) =$

$$\begin{bmatrix} F_0(e^{j\omega}) & \cdots & F_{C-1}(e^{j\omega}) \\ \vdots & \ddots & \vdots \\ F_0(e^{j\omega}W^{(D-1)}) & \cdots & F_{C-1}(e^{j\omega}W^{(D-1)}) \end{bmatrix} \tag{7}$$

for the synthesis bank. As with the CSD matrix representation, the AC representation is only defined for $|\omega|\le\pi/D$ and provides a complete and unique frequency domain representation for the corresponding set of filters. An AC matrix product of interest is defined as

$$\mathbf{P}(e^{j\omega}) = \frac{1}{D}\mathbf{F}_{AC}(e^{j\omega})\mathbf{H}_{AC}^T(e^{j\omega}). \tag{8}$$

The CSD matrix of $y[n]$, also defined through (5), is related the the CSD matrix of $x[n]$ through

$$\mathbf{S}_{yy}(e^{j\omega}) = \mathbf{P}(e^{j\omega})\mathbf{S}_{xx}(e^{j\omega})\mathbf{P}^H(e^{j\omega}), \tag{9}$$

where the superscript H denotes the conjugate transpose. The cross-CSD matrices are defined through

$$\mathbf{S}_{yx}(e^{j\omega}) = \mathbf{P}(e^{j\omega})\mathbf{S}_{xx}(e^{j\omega}) \tag{10}$$

and

$$\mathbf{S}_{xy}(e^{j\omega}) = \mathbf{S}_{xx}^H(e^{j\omega})\mathbf{P}^H(e^{j\omega}), \tag{11}$$

and are also only valid for $|\omega|\le\pi/D$. The CSD matrix of $e[n]$ is

$$\mathbf{S}_{ee}(e^{j\omega}) = \mathbf{S}_{xx}(e^{j\omega}) - \mathbf{S}_{yx}(e^{j\omega}) - \mathbf{S}_{xy}(e^{j\omega}) + \mathbf{S}_{yy}(e^{j\omega}), \tag{12}$$

which, through (9-11), is found to be a function of the AC matrices and the CSD matrix of $x[n]$.

The TAMSE defined in (2) is manipulated to find an equivalent representation as a function of $\mathbf{S}_{ee}(e^{j\omega})$ through

$$\begin{aligned} \sigma_e^2 &= \frac{1}{D}\sum_{n=0}^{D-1} R_{ee}[n,n] = R_{ee}^0[0] = \frac{1}{2\pi}\int_0^{2\pi} S_{ee}^0(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\sum_{k=0}^{D-1}\int_{-\pi/D}^{\pi/D} S_{ee}^0(e^{j\omega}W^k)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/D}^{\pi/D} \mathrm{tr}(\mathbf{S}_{ee}(e^{j\omega}))d\omega. \end{aligned} \tag{13}$$

The minimization of σ_e^2 is thus equivalent to the minimization of $\mathrm{tr}(\mathbf{S}_{ee}(e^{j\omega}))$ (since $S_{ee}^0(e^{j\omega})$ is nonnegative). A frequency domain representation of the reconstruction filters that minimize the TAMSE is found by solving the problem

$$\min_{\mathbf{F}_{AC}(e^{j\omega})} \quad \mathrm{tr}(\mathbf{S}_{ee}(e^{j\omega})) \quad \text{over } |\omega|\le\pi/D. \tag{14}$$

Defining the matrices

$$\mathbf{Q}(e^{j\omega}) = \mathbf{H}_{AC}^T(e^{j\omega})\mathbf{S}_{xx}(e^{j\omega}) \tag{15}$$

and

$$\mathbf{R}(e^{j\omega}) = \mathbf{H}_{AC}^T(e^{j\omega})\mathbf{S}_{xx}^H(e^{j\omega})\mathbf{H}_{AC}^*(e^{j\omega}), \tag{16}$$

the solution to (14) is found to be

$$\mathbf{F}_{AC,opt}(e^{j\omega}) = D\mathbf{Q}^H(e^{j\omega})\mathbf{R}^{-1}(e^{j\omega}). \tag{17}$$

This solution assumes an inverse to (16) exists.

Applying the optimal solution (17) to $\mathbf{S}_{ee}(e^{j\omega})$ as defined by (12) determines the optimal error CSD matrix

$$\begin{aligned}\mathbf{S}_{ee,opt}(e^{j\omega}) &= \mathbf{S}_{xx}(e^{j\omega}) - \mathbf{Q}^H(e^{j\omega})\mathbf{R}^{-H}(e^{j\omega})\mathbf{Q}(e^{j\omega}) \\ &= \mathbf{S}_{xx}(e^{j\omega}) - \frac{\mathbf{Q}^H(e^{j\omega})\mathbf{F}_{AC,opt}^H(e^{j\omega})}{D}. \end{aligned} \tag{18}$$

This result can be applied to (13) to find $\sigma_{e,opt}^2$.

In most cases, a numerical solution to (17) will need to be found. Solving $\mathbf{F}_{AC,opt}(e^{j\omega})$ for a selection of frequencies over $|\omega| \leq \pi/D$ will determine a discrete representation for the frequency responses of the reconstruction filters. The expression (17) can be solved for a sufficient number of frequencies to enable design of an accurate implementation. Similarly, a numerical solution will typically have to be found to determine the minimized TAMSE. The optimal CSD $S_{ee}^0(e^{j\omega})$ is found along the diagonal of (18). A sampled representation for this function can be determined along with that of $\mathbf{F}_{AC,opt}(e^{j\omega})$. An accurate numerical approximation to the integration of (13) can be calculated. The sampling density necessary for accurate representation of the synthesis filters will typically provide a sufficiently accurate representation of $S_{ee}^0(e^{j\omega})$.

The mean-squared value of $e[n]$ will generally be a periodic with period D. The calculated $\sigma_{e,opt}^2$ only provides the average MSE and the expected MSE at any specific index can vary from this value. However, since $e[n]$ is WSCS(D), the error of any D-fold decimated sequence obtained from it will have a constant MSE which can be calculated using a polyphase decomposition of the synthesis filters. More importantly, it is shown in [11] that the D-fold decimated sequences of $e[n]$ are individually optimized in the mean-squared sense by (17). This means that a high resolution model $x[n]$ (corresponding to a small T_M and large D) determines an optimal high resolution reconstruction solution as well as optimal lower resolution reconstructions that are composed of interleaved D-fold decimated subsequences of $y[n]$.

4. EXAMPLE

A periodically sampled 8th order WSS autoregressive (AR) process will be considered in this example. The process is generated by passing unit variance white Gaussian noise through the infinite impulse response filter

$$G(e^{j\omega}) = \frac{b}{1 + \sum_{k=1}^{8} a_i e^{-jk\omega}}$$

where $b = .0001$ and $\{a_1, \cdots, a_8\} = \{-6.9167, 21.8179, -40.8841, 49.7104, -40.1396, 21.0263, -6.5402, 0.9271\}$. The PSD of this AR process is shown in Fig. 2. Periodic sampling will be considered at an average rate of $1/6$th the rate of $x[n]$. While $S_{xx}(e^{j\omega})$ does significantly decay at higher frequencies, undersampling it by a factor of 6 will still cause aliasing of some of the more significant portions of its spectrum. The sampling process model will use $C = 3$ uniform samplers, each with sampling period $D = 18$ times that of $x[n]$.

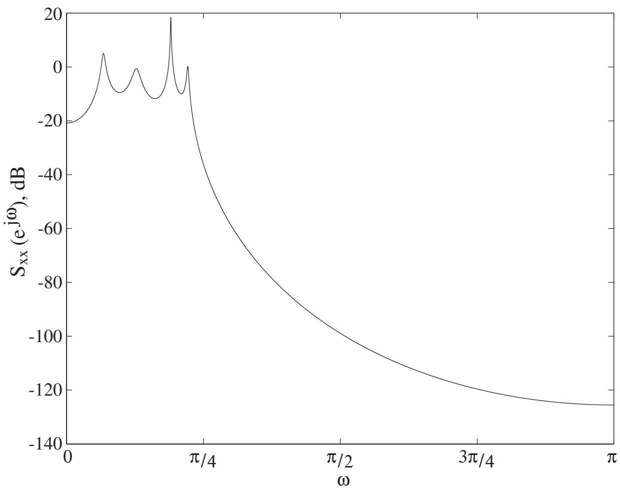

Fig. 2. PSD of 8th order AR process defined in Section 4.

The two periodic sampling patterns of Fig. 3 will be considered. Collected samples are indicated by impulse functions, while Xs indicate indexes where no sample is collected. Only one period ($D = 18$ samples) of each pattern is shown. Pattern (a) is a periodic nonuniform sampling pattern and pattern (b) is a uniform sampling pattern, which is modelled in this case by three uniformly interleaved samplers. The analysis filters corresponding to these sampling patterns are integer delays.

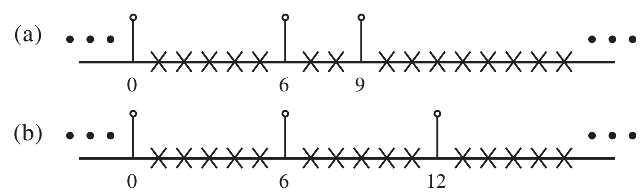

Fig. 3. Two periodic sampling patterns considered for the AR signal shown in Fig. 2.

Optimal reconstruction filters are found for both patterns through (17). The magnitude responses of these filters are shown in Fig. 4. The nonuniform sampling pattern (a) produces the left column of filters, and the uniform pattern (b) produces the right column of filters. The uniform pattern's filters are identical except for a linear phase shift corresponding to the sampler interleaving. The filters are otherwise identical to the optimal filter which would be produced using a $C = 1$ and $D = 6$ filter bank. The nonuniform pattern (a) produces three distinct reconstruction filters.

Although both sampling patterns operate at the same average rate, there is a significant difference in their performances. Numerical integration of the minimized error CSD $S_{ee}^0(e^{j\omega})$ using (18) and (13) yields errors of 2.6898% of the input signal's power for pattern (a) and 7.5759% of the input signal's power for pattern (b). This corresponds to an almost threefold reduction in the optimal TAMSE from simply shifting the timing of a single sampler. A magnified portion of the $S_{ee}^0(e^{j\omega})$ functions are shown in Fig. 5, along with the input signal PSD. While the nonuniform pattern (a) provides a significantly lower TAMSE, it is not universally better than pattern (b) for all frequencies.

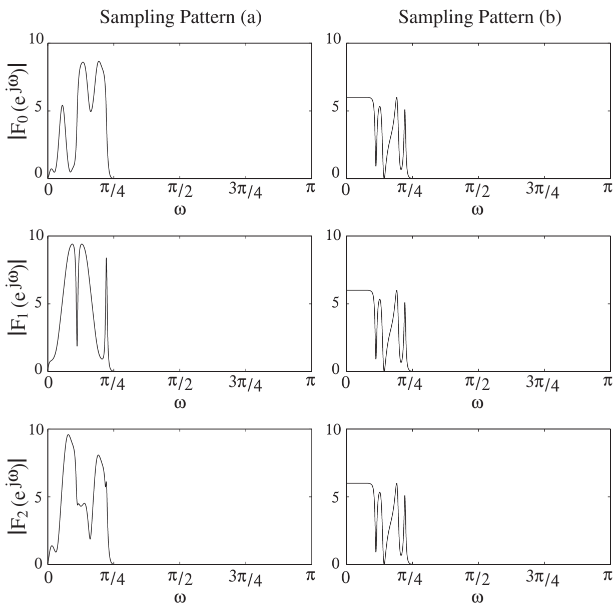

Fig. 4. Magnitude responses of optimal reconstruction filters for AR input signal in Fig. 2 using periodic sampling patterns in Fig. 3.

Previous work in periodic nonuniform sampling has considered sub-Nyquist average sampling rates for multiband signals [6, 7] that achieved PR (a sufficient sampling density in proportion to the signal's total spectral support was required). This example indicates a similar result can be obtained for undersampled signals which have segmented areas of power concentration over a contiguous span of spectral support. This paper's method can also be used to find a PR solution if a sufficient portion of $S_{xx}(e^{j\omega})$ is zero and the proper sampling pattern is selected. Selection of an optimal sampling pattern for a particular PSD will be a topic of future work.

5. CONCLUSION

This paper has presented a minimized TAMSE reconstruction for periodically sampled cyclostationary signals. A multirate filter bank was used to model a periodic sampling process and perform signal reconstruction. A high resolution discrete-time model was used in order to consider reconstruction of undersampled signals. The provided example demonstrated the effectiveness of these results and used different periodic sampling patterns to illustrate an area of interest for future investigation.

6. REFERENCES

[1] A. Papoulis, "Generalized sampling expansion," *IEEE Trans. Circuits Systems*, vol. CAS-24, pp. 652-654, Nov. 1977.

[2] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood-Cliffs, NJ: Prentice-Hall, 1993.

[3] P. P. Vaidyanathan and V. C. Liu, "Efficient reconstruction of band-limited sequences from nonuniformly decimated versions by use of polyphase filter banks," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1927-1936, Nov. 1990.

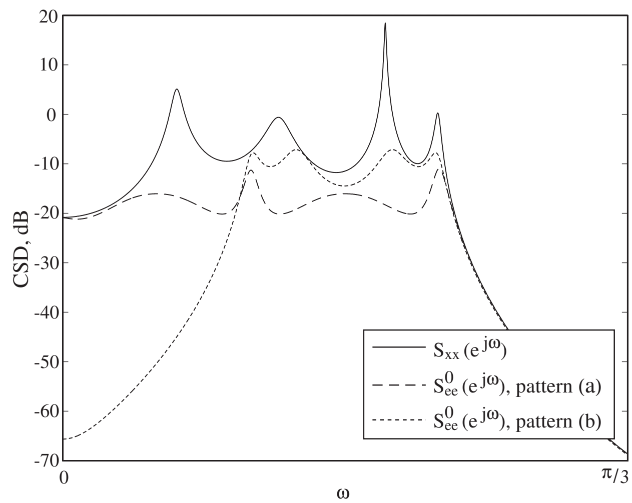

Fig. 5. Portion of PSD of $x[n]$ from Fig. 2, along with the minimized CSDs $S_{ee}^0(e^{j\omega})$ for the sampling patterns depicted in Fig. 3.

[4] P. P. Vaidyanathan and S.-M. Phoong, "Reconstruction of sequences from nonuniform samples," in *Proc. IEEE ISCAS*, Seattle, WA, 28 Apr. - 3 May, 1995.

[5] Y. C. Eldar and A. V. Oppenheim, "Filterbank reconstruction of bandlimited signals from nonuniform and generalized samples," *IEEE Trans. Sig. Proc.*, vol. 48, pp. 2864-2875, Oct. 2000.

[6] C. Herley and P. W. Wong, "Minimum rate sampling and reconstruction of signals with arbitrary frequency support," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1555-1564, July 1999.

[7] R. Venkataramani, Y. Bresler, "Optimal sub-Nyquist nonuniform sampling and reconstruction for multiband signals," *IEEE Trans. Sig. Proc.*, vol. 49, pp. 2301-2313, Oct. 2001.

[8] H. Johansson and P. Löwenborg, "Reconstruction of nonuniformly sampled bandlimited signals by means of digital fractional delay filters," *IEEE Trans. Sig. Proc.*, vol. 50, pp. 2757-2767, Nov. 2002.

[9] R. S. Prendergast, B. C. Levy, and P. J. Hurst, "Reconstruction of bandlimited periodic nonuniformly sampled signals through multirate filter banks," *IEEE Trans. Circuits Syst. I*, vol. 51, pp. 1612-1622, Aug. 2004.

[10] D. Seidner and M. Feder, "Optimal generalized sampling expansion," in *Proc. ICASSP*, Phoenix, AZ, 15-19 March, 1999.

[11] R. S. Prendergast and T. Q. Nguyen, "Minimum mean-squared error reconstruction for generalized undersampling of cyclostationary processes," submitted to *IEEE Trans. Sig. Proc.*.

[12] V. Sathe and P. P. Vaidyanathan, "Effects of multirate systems on the statistical properties of random signals," *IEEE Trans. Sig. Proc.*, pp. 131146, Jan. 1993.

[13] S. Ohno and H. Sakai, "Optimization of filter banks using cyclostationary spectral analysis," *IEEE Trans. Sig. Proc.*, vol. 44, pp. 2718-2725, Nov. 1996.