WEIGHTED MEDIAN FILTERS FOR MULTICHANNEL SIGNALS

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ABSTRACT

This paper focuses on extending the weighted median for use with multidimensional (multichannel) signals. Sorting multicomponent (vector) values and selecting the middle value is not well defined as in the scalar case. This paper introduces two median based multivariate filtering structures inspired by ML estimates of location in multivariate spaces. Unlike Astola's weighted vector median filter, the multichannel weighted median filter structures introduced in this paper are able to exploit the spatial and cross-channel correlations embedded in the data. Adaptive optimization algorithms for the filters are derived. The effectiveness of these algorithm is shown through image and array processing experiments.

1. INTRODUCTION

The formulation of weighted median filtering in multivariate domains can be found through the minimization of a weighted cost function that takes into account the multicomponent nature of the data. The first approach was taken by Astola et.al where the original definition of Vector WM filtering requires the extension of the original WM filter definition as follows (Astola 1990 [1]). The filter input vector is denoted as $\mathbf{X} = [\vec{X}_1\ \vec{X}_2\ \ldots\ \vec{X}_N]^T$, where $\vec{X}_i = [X_i^1\ X_i^2\ \ldots\ X_i^M]^T$ is the ith M-variate sample in the filter window. The filter output is $\vec{Y} = [Y^1\ Y^2\ \ldots\ Y^M]^T$. Recall that the weighted median of a set of 1-dimensional samples X_i $i = 1, \ldots, N$ is given by

$$Y = \arg\min_{\beta} \sum_{i=1}^{N} |W_i||\mathrm{sgn}(W_i)X_i - \beta|. \quad (1)$$

Extending this definition to a set of M-dimensional vectors $\vec{X}_i$ for $i = 1, \ldots, N$ leads to

$$Y = \arg\min_{\vec{\beta}} \sum_{i=1}^{N} |W_i|\|\vec{\beta} - \vec{S}_i\|_p\ . \quad (2)$$

where $\vec{Y} = [Y^1,\ Y^2, \ldots,\ Y^M]^T$, $\vec{S}_i = \mathrm{sgn}(W_i)\vec{X}_i$, and $\|\cdot\|_p$ is the L_p norm defined as

$$\|\vec{\beta} - \vec{S}_i\| = \left((\beta^1 - S_i^1)^p + (\ldots + (\beta^M - S_i^M)^p\right)^{\frac{1}{p}}. \quad (3)$$

The vector weighted median thus requires N scalar weights, with one scalar weight assigned per each input vector sample. Unlike the 1-dimensional case, $\vec{Y}$ is not generally equal in value to one of the $\vec{S}_i$. Indeed, there is no closed-form solution for $\vec{Y}$. Moreover, solving (2) involves a minimization problem in a M-dimensional space that can be computationally expensive. To overcome these difficulties, a suboptimal solution for (2) is found if $\vec{Y}$ is restricted to be one of the signed samples $\vec{S}_i$ as in

$$\vec{Y} = \arg\min_{\vec{\beta}\in\{\vec{S}_i\}} \sum_{i=1}^{N} |W_i|\|\vec{\beta} - \vec{S}_i\|_p\ . \quad (4)$$

Several optimization algorithms for the design of the weights have been developed. Despite their existence, weighted vector medians have not significantly spread beyond image smoothing applications.

In the following, more general vector median filter structures are presented. These structures are capable of capturing and exploiting the spatial and cross-channel correlations embedded in the data. They can also be adapted to admit positive and negative weights using sign coupling.

2. WEIGHTED MULTICHANNEL MEDIAN FILTERING STRUCTURES

The multivariate filtering structure is derived from the ML estimate of location, this time in a multivariate signal space. Consider a set of independent but not identically distributed vector valued samples, each obeying a joint Gaussian distribution with the same location parameter $\vec{\mu}$,

$$f(\vec{X}_i) = \frac{1}{(2\pi)^{\frac{M}{2}}|\mathbf{C}_i|^{\frac{1}{2}}}\ e^{-\frac{1}{2}(\vec{X}_i-\vec{\mu})^T\mathbf{C}_i^{-1}(\vec{X}_i-\vec{\mu})}, \quad (5)$$

where $\vec{X}_i$ and $\vec{\mu}$ are all M-variate column vectors, and $\mathbf{C}_i^{-1}$ is the $M \times M$ cross-channel correlation matrix of the sample

$\vec{X}_i$. The Maximum Likelihood estimation of location $\vec{\mu}$ can be derived as

$$\vec{\mu} = \left(\sum_{i=1}^{N} \mathbf{C}_i\right)\left(\sum_{i=1}^{N} \mathbf{C}_i^{-1}\vec{X}_i\right). \tag{6}$$

As in the univariate case, a general multivariate filtering structure results from the maximum likelihood estimator as

$$Y = \sum_{i=1}^{N} \mathbf{W}_i^T X_i, \tag{7}$$

where $\mathbf{W}_i{}^T = \left(\sum_{i=1}^{N} \mathbf{C}_i\right)\mathbf{C}_i^{-1}$.

An example of an optimal filter design algorithm for this linear filtering structure is shown by Robinson (1983) [2]. It presents one inconvenience: the size of the weight matrix. For instance, to filter a 3-channel color image using a 5x5 window requires the optimization of 225 weights. Alternative filter structures requiring lesser weights are needed. The following approach provides such implementation.

2.1. Weighted Multichannel Median (WMM) Filter I

In most multichannel applications, the signals from subchannels are often correlated in stationary or at least quasi-stationary structures. In these cases it can be assumed that

$$\mathbf{C}_i^{-1} = q_i\mathbf{C}^{-1}. \tag{8}$$

The corresponding MLE is then

$$\vec{\mu} = \left(\sum_{i=1}^{N} q_i\mathbf{C}^{-1}\right)^{-1}\left(\sum_{i=1}^{N} q_i\mathbf{C}^{-1}\vec{X}_i\right). \tag{9}$$

In consequence, the filtering structure can be formulated as

$$\vec{Y} = \sum_{i=1}^{N} V_i\mathbf{W}^T\vec{X}_i \tag{10}$$

where V_i is the (time/spatial) weight applied to the ith vector sample in the observation window and $\mathbf{W}$ is the cross-channel weight matrix exploiting the correlation between the components of a sample. The filter thus consists of $M^2 + N$ weights. In the example of a RGB image with a 5×5 window, the number of weights would be reduced from 225 to $3^2 + 25 = 34$.

Even though it is mathematically intractable to derive a similar result as in (10) from a multivariate Laplacian distribution, it is still possible to define a nonlinear multivariate filter by direct analogy by replacing the summations in (10) with median operators. This filter is referred to as the Weighted Multichannel Median (WMM) and is defined as follows (Li et al. (2004) [3]).

$$\vec{Y} = \text{MEDIAN}(|V_i| \diamond \text{sgn}(V_i)\vec{Q}_i \mid_{i=1}^{N}), \tag{11}$$

where

$$\vec{Q}_i = \begin{bmatrix} \text{MEDIAN}(|W^{j1}| \diamond \text{sgn}(W^{j1})X_i^j \mid_{j=1}^{M}) \\ \vdots \\ \text{MEDIAN}(|W^{jM}| \diamond \text{sgn}(W^{jM})X_i^j \mid_{j=1}^{M}) \end{bmatrix} \tag{12}$$

is an M-variate vector. As it was stated before, there is no unique way of defining even the simplest median over vectors, in consequence, the outer median in (11) can have several different implementations. Due to its simplicity and ease of mathematical analysis, a suboptimal implementation where the outer median in (11) is replaced by a vector of marginal medians can be used. Thus, the Marginal Weighted Multichannel Median (Marginal WMM) is defined as

$$\vec{Y} = \begin{bmatrix} \text{MED}(|V_i| \diamond \text{sgn}(V_i)Q_i^1 \mid_{i=1}^{N}) \\ \vdots \\ \text{MED}(|V_i| \diamond \text{sgn}(V_i)Q_i^M \mid_{i=1}^{N}) \end{bmatrix}, \tag{13}$$

where $Q_i^l = \text{MED}(|W^{jl}| \diamond \text{sgn}(W^{jl})X_i^j \mid_{j=1}^{M})$ for $l = 1, \ldots, M$.

2.2. Weighted Multichannel Median (WMM) Filter II

There are some applications where the initial assumption about stationarity stated in (8) may not be appropriate. The need of a simpler filtering structure remains, and this is why a more general structure for median filtering of multivariate signals is presented. In such case replace (8) by

$$\begin{aligned} \mathbf{C}_i^{-1} &= \text{diag}(q_i)\mathbf{C}^{-1} \qquad (14) \\ &= \begin{bmatrix} q_i^1 C^{11} & \cdots & q_i^1 C^{1M} \\ \vdots & \ddots & \vdots \\ q_i^M C^{M1} & \cdots & q_i^M C^{MM} \end{bmatrix}. \end{aligned}$$

In this case, the cross-channel correlation is not stationary, and the q_i^j represent the correlation between components of different samples in the observation window. The linear filtering structure reduces to

$$\begin{aligned} Y &= \sum \text{diag}(\vec{V}_i)\mathbf{W}\vec{X}_i \qquad (15) \\ &= \sum \begin{bmatrix} V_i^1 \sum W^{j1}X_i^j \\ \vdots \\ V_i^M \sum W^{jM}X_i^j \end{bmatrix}, \qquad (16) \end{aligned}$$

where V_i^l is the weight reflecting the influence of the lth component of the ith sample in the lth component of the output. The weights W^{ij} have the same meaning as in the WMM filter I. Using the same analogy used in the previous case, a more general weighted multichannel median filter structure can be defined as

$$\vec{Y} = \begin{bmatrix} \text{MEDIAN}(|V_i^1| \diamond \text{sgn}(V_i^1)\text{MEDIAN}(|W^{j1}|) \\ \diamond\, \text{sgn}(W^{j1}) X_i^j |_{j=1}^M)|_{i=1}^N \\ \vdots \\ \text{MEDIAN}(|V_i^M| \diamond \text{sgn}(V_i^M)\text{MEDIAN}(|W^{jM}|) \\ \diamond\, \text{sgn}(W^{jM}) X_i^j |_{j=1}^M)|_{i=1}^N \end{bmatrix}. \quad (17)$$

This structure does not require suboptimal implementations like the previous one. The number of weights increases, but is still significantly small. For the image filtering example, the number of weights will be $M \times (N + M) = 84$.

In the following section, optimal adaptive algorithms for the structures in (11) and (17) are defined.

3. FILTER OPTIMIZATION

LMA algorithms for the optimization for the weights of both filtering structures were developed. The observed process is denoted as $\vec{X}(n)$, the desired output is $\vec{D}(n)$. The results of the process are shown below.

3.1. Optimization for the WMM Filter I

Assume that the time/spatial dependent weight vector is $\vec{V} = [V_1\ V_2\ \ldots\ V_N]^T$, and the cross-channel weight matrix is $\mathbf{W} = \left[W^{ij}\right]|_{i,j=1}^M$. The adaptive algorithm for $\vec{V}$ is as follows,

$$V_i(n+1) = V_i(n) + \mu_v \text{sgn}(V_i(n))\vec{e}^T(n)\vec{G}_i^{\hat{D}}(n), \quad (18)$$

where $\vec{G}_i^{\hat{D}} = [G_i^{\hat{D}^1}\ \ldots\ G_i^{\hat{D}^M}]^T$ and $G_i^{\hat{D}^l} = \text{sgn}(\text{sgn}(V_i)Q_i^l - \hat{D}^l)$ for $l = 1, \ldots, M$.

The updates for $\mathbf{W}$ are as follows

$$\begin{aligned} W^{st}(n+1) &= W^{st}(n) + \quad (19) \\ &\quad \mu_w \text{sgn}(W^{st}(n))e^t(n)(\vec{V}^T(n)\vec{A}^s(n)), \end{aligned}$$

where $\vec{A}^s = [A_1^s\ A_2^s\ \ldots\ A_N^s]^T$, and $A_i^s = \delta(\text{sgn}(V_i)Q_i^l - \hat{D}^l)\text{sgn}(\text{sgn}(W^{st})X_i^s - Q_i^t)$ for $i = 1, \ldots, N$, where $\delta(x) = 1$ for $x = 0$ and $\delta(x) = 0$ otherwise.

3.2. Optimization of the WMM Filter II

The updates for $\mathbf{V}$ result in

$$V_s^t(n+1) = V_s^t(n) + \mu_v e^t(n)\text{sgn}(V_s^t(n))G_s^{\hat{D}^t} \quad (20)$$

On the other hand, the updates for $\mathbf{W}$ are given by:

$$\begin{aligned} W^{st}(n+1) &= W^{st}(n) + \mu_w \text{sgn}(W^{st}(n))e^t(n) \quad (21) \\ &\quad (\vec{V^l}^T(n)\vec{A}^{st}(n)), \end{aligned}$$

that is basically the same as (20) with the difference that $\mathbf{V}$ is now a matrix and $\vec{A}^{st} = [\delta(\text{sgn}(V_i^t)Q_i^t - \hat{D}^t)\text{sgn}(\text{sgn}(W^{st})X_i^s - Q_i^t)|_{i=1}^N]$.

4. SIMULATIONS

4.1. Color image denoising

A RGB color image contaminated with 10% correlated salt-and-pepper noise is processed by the WVM filter, and the marginal WMM filter separately. The observation window is set to 3×3 and 5×5. The optimal weights for the marginal WMM filter are obtained first by running the LMA algorithm derived above over a small part of the corrupted image. The same section of the noiseless image is used as a reference. A similar procedure is repeated to optimize the weights of the WVM filter. The resulting weights are then passed to the corresponding filters to denoise the whole image. The filter outputs are depicted in Figure 1. As a measure of the effectiveness of the filters, the mean absolute error and the Peak signal-to-noise ratio (PSNR) of the outputs were calculated for each filter, the results are summarized in Table 1.

Table 1 shows that the marginal WMM filter outperforms the WVM filter by a factor of 3 in MAE, or 8-11dB in PSNR. Also, the output of the WVM filter is visually less pleasant with many unfiltered outliers. Notice that the output of the marginal WMM filter with the smaller window preserves more image details and has a better PSNR though the MAE in the two cases are roughly the same.

Table 1. Average MAE and PSNR of the output images.

Filter	MAE		PSNR (dB)	
	3×3	5×5	3×3	5×5
Noisy signal	0.1506		14.66	
WVM	0.0748	0.0732	23.41	27.74
marginal WMM	0.0248	0.0247	32.26	32.09

4.2. Array Processing with the WMM filter II

To test the effectiveness of the WMM filter II, a simple array processing problem with real-valued signals is used. The system shown in Figure 2 is implemented. It consists of a 3 element array and 3 sources in the farfield of the array transmitting from different directions and at different frequencies as indicated in the figure. The goal is to separate the signals from all sources using the array in the presence of alpha stable noise. In order to do so, a WVM filter, a marginal WMM filter and a WMM filter II all with a window size of 25 are used. The filters are optimized using the algorithms described earlier in this paper, with a reference signal whose components are noiseless versions of the signals emitted by the sensors.

The results obtained are summarized in Figure 3 and Table 2. Figure 3 shows that the WMM filter II is able to extract the desired signals from the received signals at the sensors successfully. The WVM filter and the marginal WMM

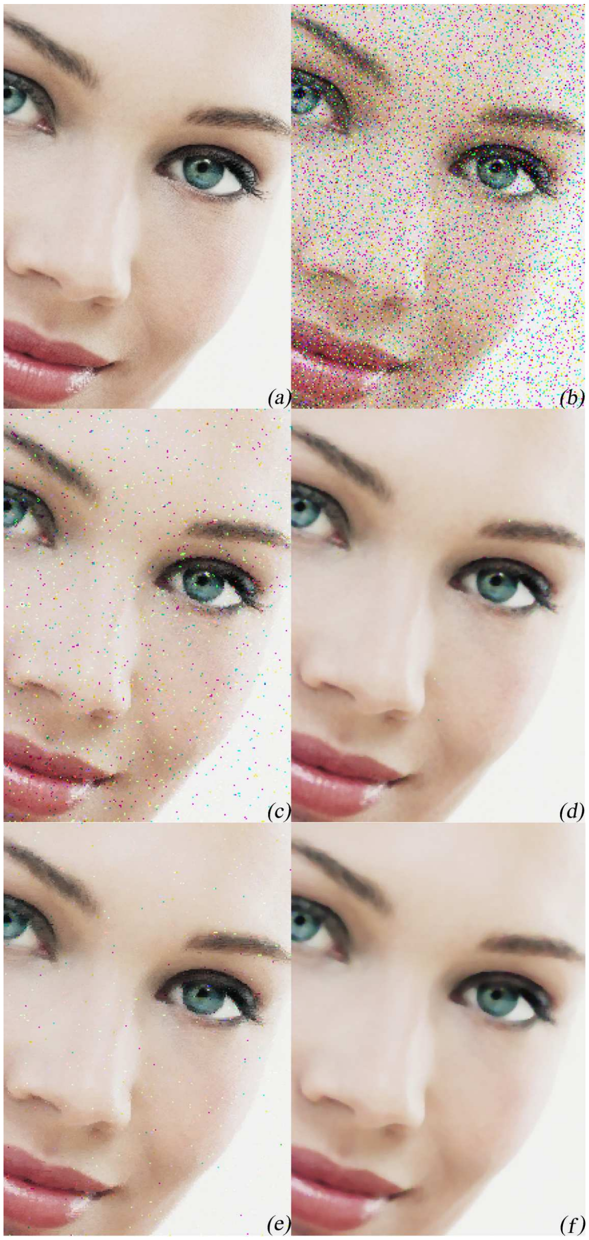

Fig. 1. Multivariate medians for color images in salt-and-pepper noise, $\mu = 0.001$ for the WVM, $\mu_v, \mu_w = 0.05$ for the marginal WMM. Noiseless image, contaminated image, WVM and marginal WMM with 3×3 window, WVM and marginal WMM with 5×5 window.

filter I are unable to do so. Linear filters were implemented with adaptive algorithms to optimize them for this problem but the impulsiveness of the noise made the optimization algorithms diverge.

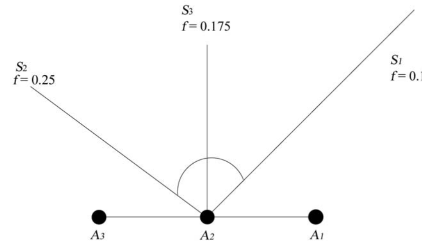

Fig. 2. Array of Sensors

Table 2. Average MAE of the output signals.

Filter	MAE
Noisy signal	0.8248
WVM	0.6682
Marginal WMM I	0.5210
WMM II	0.3950

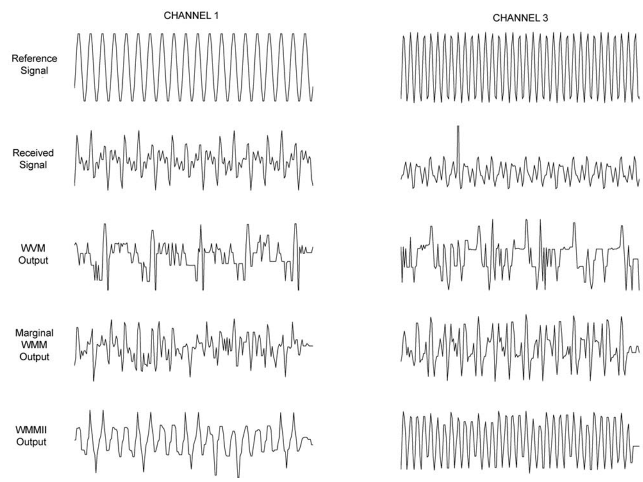

Fig. 3. Array processing with multivariate medians. The columns correspond to channels one and three and the rows represent: the reference signal, the signal received at the sensors, the output of the WVM filter, the output of the marginal WMM filter and the output of the WMM filter II.

5. REFERENCES

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