RECONSTRUCTION OF COUPLING PROFILES FOR SCATTERING MEDIA BY THE SCHUR ALGORITHM COMBINED WITH AN EXTRAPOLATION METHOD

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ABSTRACT

We demonstrate an efficient inverse scattering algorithm for reconstructing the coupling profiles or reflection coefficients of a discrete layered medium. The method, called the Schur algorithm combined with an extrapolation method, is based on solving the coupled-mode differential equation in a layer-peeling procedure with a simple extrapolation scheme. In order to reduce the estimation errors caused by discretization of an inhomogeneous medium, we analyze the error propagation of inverse scattering with the Schur algorithm (layer-peeling method) and propose the extrapolation scheme. Numerical examples are provided that compare the Schur algorithm combined with an extrapolation method to the general Schur algorithm in a coarse discretization environment. The comparison shows that the proposed method produces a more accurate reconstruction with a lower order of complexity.

1. INTRODUCTION

The Schur algorithm (layer-peeling method) for inverse scattering problems is an efficient method to identify the coupling profiles of a layered wave propagation medium such as transmission-lines, earth, vocal tract, and fiber Bragg gratings [1,2,3].

When scattering data are contaminated by noise, stochastic signal processing techniques have been applied [4,5]. Iterative layer-peeling [6] has also been reported that adaptively reconstructs the input waveforms to reduce errors. These methods are related to eliminate the errors caused by noise with a fine discretization. On the other hand, our algorithm eliminates errors caused by coarse discretizaion of the medium. In general, to achieve a precise approximation using the Schur algorithm, we usually wish to have fine discretization, which in turn results in a large memory size and is a time consuming operation. In a coarse discretization environment, the Schur algorithm will return inaccurate results. The main source of the inaccuracy with a noise free assumption is global discretization errors that accumulate during the peeling process. In order to eliminate global discretization errors, we introduce the Schur algorithm combined with an extrapolation method, which produces a high order approximation with a minimal computational cost in a coarse discretization environment.

In order to apply an extrapolation scheme to a Schur algorithm, we first analyze the global discretization error propagation of the Schur algorithm analytically and show the upper-bound. Then, based on the error analysis, we obtain the asymptotic expansion that reveals the global discretization error of approximated coupling profiles as a polynomial of infinite degree with a discretizing step size of h. The extrapolation scheme that cancells global discretization errors through the layers is determined from the asymptotic expansion. Numerical simulations and a complexity analysis show that a high degree of accuracy is achieved for the proposed method with a notably lower complexity than for the general Schur algorithm. This efficient inverse scattering algorithm can be applied to identification for scattering media such as the estimation of transmission line parameters, nondestructive testing, vocal tract modelling, and fiber Bragg grating filter synthesis.

2. INVERSE SCATTERING WITH SCHUR ALGORITHM

Let x be a spatial variable and t be time. Scattering media are described by a two-component wave system coupled mode differential equation [1]

$$\frac{d}{dx}\begin{bmatrix} w_R(t,x) \\ w_L(t,x) \end{bmatrix} = \begin{bmatrix} -\frac{d}{dt} & -q(x) \\ -q(x) & \frac{d}{dt} \end{bmatrix} \begin{bmatrix} w_R(t,x) \\ w_L(t,x) \end{bmatrix} \qquad (1)$$

where $w_R(t,x)$ and $w_L(t,x)$ are the right and left propagating wave, and $q(x)$ is the coupling function. Inverse scattering identifies the coupling function $q(x)$, which characterizes the scattering medium.

Consider a discrete layered medium that is composed of N uniform loss-less layered sections with the assumption that the medium was initially quiescent. Discretization is performed in such a way that the travel time for propagating waves from one end of each section to the other end of the section is a unit time of h. After straightforward discretization of (1), the recurrent relation for the propagating waves becomes, for $i = 0, 1, \ldots, N-1$ [1],

$$\begin{bmatrix} w_R(t,x_{i+1}) \\ w_L(t,x_{i+1}) \end{bmatrix} = \frac{1}{\pi_i}\begin{bmatrix} z^{-\frac{1}{2}} & 0 \\ 0 & z^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} 1 & -hq(x_i) \\ -hq(x_i) & 1 \end{bmatrix}\begin{bmatrix} w_R(t,x_i) \\ w_L(t,xi) \end{bmatrix} \qquad (2)$$

, where $z^{-\frac{1}{2}}$ and $z^{\frac{1}{2}}$ are the unit time-delay and the unit time-advance operators defined by $z^{-\frac{1}{2}}w(t,x_i) = w(t-h,x_i)$ and $z^{\frac{1}{2}}w(t,x_i) = w(t+h,x_i)$ respectively, and the transmission coefficient π_i is defined by $\sqrt{1-(hq(x_i))^2}$. We now let $t = \tau$ and $2h \equiv \Delta$, where τ indicates the time of the first appearing right propagating wave as the input to the i-th section, and Δ represents the time in which the injected wave from the i-th section to the $i+1$-th section is reflected after undergoing scattering. The coupling or reflection coefficient at $x = x_i$ is computed as the ratio of the first impinging wave $w_R(\tau, x_i)$ and the first reflected wave

$w_L(\tau, x_i)$, $hq(x_i) = w_L(\tau, x_i)/w_R(\tau, x_i)$[1]. This interpretation is based on the causality principle, which states that the front edge of the impulse response of a medium is proportional to the coupling profile at the begging of the i-th section [1]. The reconstructed coupling coefficient of the i-th section is used to find the coupling coefficient of the $i+1$-th section using a direct solution to the coupled-mode differential equation (2). Thus, the coupling coefficient of the $i+1$-th section becomes

$$hq(x_{i+1}) = \frac{w_L(\tau+\frac{\Delta}{2}, x_{i+1})}{w_R(\tau+\frac{\Delta}{2}, x_{i+1})} = \frac{w_L(\tau+\Delta, x_i) - hq(x_i) w_R(\tau+\Delta, x_i)}{w_R(\tau, x_i) - hq(x_i) w_L(\tau, x_i)} \quad (3)$$

The set of equations (2) and (3) for $i = 0, 1, \ldots, N-1$ represents the Schur algorithm (layer-peeling method) for inverse scattering [1,2]. We designate (3), the recursion formula for the coupling coefficient, as the Schur recursion formula.

3. SCHUR AND EXACT RECURSION FORMULAE

Assuming that $w_R(t,x), w_L(t,x)$ and $q(x)$ are continuous and differentiable function that satisfy the Lipschitz condition [7] for $x \geq 0$ and $t \geq 0$, we define the derivatives for $w_R(t,x)$ and $w_L(t,x)$ based on (1)

$$\begin{aligned} (w_R(t,x))^{(k)} &\equiv (\tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial t})^k w_R(t,x) \\ (w_L(t,x))^{(k)} &\equiv (\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial t})^k w_L(t,x) \end{aligned}$$

, for $k = 1, 2, \ldots$. Specifically, $(w_R(t,x))^{(1)} = -q(x) w_L(t,x)$ and $(w_L(t,x))^{(1)} = -q(x) w_R(t,x)$. Now, we assume that the scattering data at $x = x_i$ are exact and our concern now shift to computing the coupling coefficient at $x = x_{i+1}$. Our goal is to derive an appropriate series expansion for the Schur recursion formula in order to analyze the error behavior of the Schur algorithm.

The nominator of the Schur recursion formula (3) can be interpreted as the first order Taylor series for $w_L(\tau + \frac{\Delta}{2}, x_i + h)$ about $\tau + \Delta$ and x_i. The denominator can also be interpreted as the first order Taylor series for $w_R(\tau + \frac{\Delta}{2}, x_i + h)$ about τ and x_i, which yields

$$\frac{w_L(\tau+\frac{\Delta}{2}, x_{i+1})}{w_R(\tau+\frac{\Delta}{2}, x_{i+1})} = \frac{w_L(t+\Delta, x) + h(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}) w_L(t+\Delta, x)}{w_R(t,x) + h(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) w_R(t,x)} \Big|_{t=\tau, x=x_i}. \quad (4)$$

The formula $hq(x_i) = w_L(\tau, x_i)/w_R(\tau, x_i)$, computing the coupling coefficient at $x = x_i$, can also be interpreted as the first order Taylor series for $w_L(\tau - \frac{\Delta}{2}, x_{i+1})$ about τ and x_i, which leads to $w_L(\tau - \frac{\Delta}{2}, x_{i+1}) = w_L(\tau, x_i) + h(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}) w_L(t,x)|_{t=\tau, x=x_i} = 0$, where $w_L(\tau - \frac{\Delta}{2}, x_{i+1}) = 0$ is based on the causal propagation of the wave in the medium. Since $(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}) w_L(t,x)$, as shown in (1), equals $-q(x) w_R(t,x)$, we obtain the relation of $hq(x_i) = w_L(\tau, x_i)/w_R(\tau, x_i)$.

In order to analyze (4), we develop an expression for the r.h.s. of (4) in terms of $w_R(t,x)$ and $w_L(t,x)$. We suppose the partial Taylor series expansion of $w_L(t+\Delta, x)$ about t

$$w_L(t+\Delta, x) = w_L(t,x) + \sum_{k=1}^{\infty} \frac{(2h)^k}{k!} (\frac{\partial}{\partial t})^k w_L(t,x). \quad (5)$$

Then we substitute (5) into the r.h.s. of (4) and expand the Schur recursion formula (4) to a series form by long division. Then, (4) leads to the relation

$$\begin{aligned} hq(x_{i+1}) = hq(x_i) + \sum_{k=1}^{\infty} \frac{h^k}{k!} \sum_{m=0}^{k} \frac{k!}{m!} \Big(\frac{-(w_R(t,x))^{(1)}}{w_R(t,x)} \Big)^{k-m} \\ \times L_{m-1}^{m} \frac{w_L(t,x)}{w_R(t,x)} \Big|_{t=\tau, x=x_i} \end{aligned} \quad (6)$$

where the coupling coefficient computed by the Schur algorithm is iterated. The differential operator L_{m-1}^m which operates on $w_L(t,x)$ is defined by $L_{m-1}^m \equiv \sum_{j=m-1}^{m} C_j^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)^{m-j} \left(2 \frac{\partial}{\partial t} \right)^j$, where $L_{m-1}^m = 1$ when $m \leq 0$ and $C_j^m = \frac{m!}{(m-j)!j!}$.

In principle, if we are able to analyze the Schur algorithm using the first order Taylor series expansion, we can obtain the exact or true recursion formula from an infinite order Taylor series expansion. We define the exact coupling function by $\hat{q}(x)$ and the exact scattering data at $x \geq x_{i+1}$ by $\hat{w}(t,x)$. The exact coupling function multiplied by h at $x = x_i$, $h\hat{q}(x_i)$ is determined by the causal propagation of the wave with the infinite order Taylor series expansion for $\hat{w}_L(\tau - \frac{\Delta}{2}, x_{i+1})$ about τ and x_i, which yields $\hat{w}_L(\tau - \frac{\Delta}{2}, x_{i+1}) = 0 = w_L(t,x) + h(w_L(t,x))^{(1)} + \sum_{n=2}^{\infty} \frac{h^n}{n!} (w_L(t,x))^{(n)}$. Thus, we obtain

$$h\hat{q}(x_i) = \frac{w_L(t,x) + \sum_{n=2}^{\infty} \frac{h^n}{n!} \left(w_L(t,x) \right)^{(n)}}{w_R(t,x)} \Big|_{t=\tau, x=x_i} \equiv \frac{\overline{w}_L(\tau, x_i)}{w_R(\tau, x_i)}. \quad (7)$$

The exact coupling function multiplied by h at $x = x_{i+1}$ equals

$$h\hat{q}(x_{i+1}) = \frac{\hat{w}_L(t+\frac{\Delta}{2}, x+h) + \sum_{n=2}^{\infty} \frac{h^n}{n!} \left(\hat{w}_L(t+\frac{\Delta}{2}, x+h) \right)^{(n)}}{\hat{w}_R(t+\frac{\Delta}{2}, x+h)} \Big|_{t=\tau, x=x_i}. \quad (8)$$

We express the r.h.s. of (8) in terms of $w_R(t,x)$ and $w_L(t,x)$ using an infinite order Taylor series expansion, and expand the r.h.s. to a series form by applying long division, leading to

$$\begin{aligned} h\hat{q}(x_{i+1}) = h\hat{q}(x_i) + \sum_{k=1}^{\infty} \frac{h^k}{k!} \sum_{m=0}^{k} \frac{k!}{m!} \Big(\frac{-(w_R(t,x))^{(1)}}{w_R(t,x)} \Big)^{k-m} \\ \times L_{m-1}^{m} \frac{\overline{w}_L(t,x)}{w_R(t,x)} + \sum_{l=2}^{\infty} \frac{h^l}{l!} \Phi_{l-1}(t,x) \Big|_{t=\tau, x=x_i} \end{aligned} \quad (9)$$

, where Φ_{l-1} is a function of $\overline{w}_L(t,x)/w_R(t,x)$. The explicit functional form of Φ_{l-1} need not be calculated to derive the asymptotic expansion [8].

Since the determined input and measured output data at the surface of the medium, in practice, are the only exact scattering data that the Schur algorithm starts with, the assumption that the scattering data at $x = x_i$ are exact is changed to the assumption that the scattering data at $x = 0$ are exact. Therefore, the reconstructing coupling profile using the Schur algorithm is generalized as an initial value problem. To illustrate the initial value problem conjugated with the extrapolation method, we assume that a fixed step size of h_c, where h_c represents the coarse discretization step size, and the subdivided step size $h = h_j = h_c/g_j$ for $j = 0, 1, \ldots$, where g_j is defined as 2^j, which denotes the subdivision. The relation $g_j = 2^j$ permits the extrapolation method to proceed at a minimal computational cost [7]. We write the approximated h_c-step coupling coefficient for the Schur algorithm as $h_c q_i$. We also write the multi-step coupling coefficient, h_j-step coupling coefficients, as $\frac{h_c}{2_j} q_i^{<2^j>}$, where the superscript on the coupling function denotes the subdivision.

In order to derive an asymptotic expansion in an easy to follow manner, the normalized Schur recursion formula can be obtained by dividing the $h = h_j$ by both sides of (6). Then, we define the initial value problem for the normalized Schur recursion formula as

$$q_{i+1} = q_i + h\Theta\big(\frac{w_{L,x}^t}{h w_{R,x}^t}, w_{R,x}^t; h\big)\big|_{t=\tau, x=x_i}, \; q_0 = 0 \quad (10)$$

$$\Theta\big(\frac{w_{L,x}^t}{h w_{R,x}^t}, w_{R,x}^t; h\big) \equiv \sum_{k=1}^{\infty} \theta_k\big(\frac{w_{L,x}^t}{h w_{R,x}^t}, w_{R,x}^t; h\big) \quad (11)$$

$$\begin{aligned} \theta_k\big(\frac{w_{L,x}^t}{h w_{R,x}^t}, w_{R,x}^t; h\big) \equiv \frac{h^{k-1}}{k!} \sum_{m=0}^{k} \frac{k!}{m!} \Big(\frac{-(w_{R,x}^t)^{(1)}}{w_{R,x}^t} \Big)^{k-m} \\ \times L_{m-1}^{m} \frac{w_{L,x}^t}{h w_{R,x}^t} \end{aligned} \quad (12)$$

where $w_{L,x}^t$ and $w_{R,x}^t$ are approximated scattering data for the Schur algorithm. Using the same approach, the initial problem for the normalized exact recursion formula can be represented as

$$\hat{q}(x_{i+1}) = \hat{q}(x_i) + h\Theta\big(\tfrac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}, \hat{w}_R(t,x); h\big) + \tfrac{h^2}{2!}\hat{\Phi}_1(t,x) + O(h^3)|_{t=\tau, x=x_i},\ \hat{q}(0)=0 \tag{13}$$

where $\hat{\Phi}(t,x)$ is a function of $\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}$. The normalized exact recursion formula (13) suggests that the coefficients of the h^k degree term for $k = 1, 2, \dots$ are composed of true values, which implies that the coefficients of h^k degree term are independent of the step size $h = h_j$.

4. ERROR ANALYSIS AND ASYMPTOTIC EXPANSION

The error bound for the normalized Schur recursion formula is obtained by comparing (10) and (13). The goal is to derive an error bound in the form

$$q_i = \hat{q}(x_i) + h^r u(\tau, x_i) + O(h^p) \tag{14}$$

where $r < p$ and $u(\tau, x_i)$ are independent of the step size h. First, we define the error terms by $e_{i+1} \equiv q_{i+1} - \hat{q}(x_{i+1})$ and $e_i \equiv q_i - \hat{q}(x_i)$. Subtracting (13) from (10) yields

$$e_{i+1} = e_i + h\underbrace{\Big[\Theta\big(\tfrac{w^t_{L,x}}{hw^t_{R,x}}, w^t_{R,x}; h\big) - \Theta\big(\tfrac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}, \hat{w}_R(t,x); h\big)\Big]}_{=(*)} - \tfrac{h^2}{2!}\hat{\Phi}_1(t,x) + O(h^3)|_{t=\tau, x=x_i}. \tag{15}$$

Defining $(*)$ as the derivative about $\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}$ reduces the $(*)$ in (15) to s single differential equation [8]

$$(*) = \Big[\frac{\partial\theta_1\big(\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}, \hat{w}_R(t,x); h\big)}{\partial\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}}|_{t=\tau, x=x_i}\Big] e_i + O(e_i^2). \tag{16}$$

Now, we define $u(t,x)$ by the partial differential equation

$$u'(t,x) \equiv \frac{\partial\theta_1\big(\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}, \hat{w}_R(t,x); h\big)}{\partial\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}} u(t,x) - \frac{1}{2!}\hat{\Phi}_1(t,x) \tag{17}$$

with the initial condition $u(0,0) = 0$. Then, $u(\tau + \frac{\Delta}{2}, x_{i+1})$ can be expanded as below

$$u(\tau + \tfrac{\Delta}{2}, x_i + h) = u(\tau, x_i) + hu'(t,x) + O(h^2)\ |_{t=\tau, x=x_i}\ . \tag{18}$$

Subtracting $h\times$(18) from (15) and letting $e_{i+1} - hu(\tau + \frac{\Delta}{2}, x_{i+1}) \equiv w_{i+1}$ and $e_i - hu(\tau, x_i) \equiv w_i$ produces

$$w_{i+1} = w_i + h\underbrace{\frac{\partial\theta_1\big(\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}, \hat{w}_R(t,x); h\big)}{\partial\frac{\overline{w}_L(t,x)}{h\hat{w}_R(t,x)}}}_{=(**)} w_i + O(h^3). \tag{19}$$

Assuming that $|(**)| \le M$, (19) yields

$$|w_{i+1}| \quad \le \quad (1 + hM)|w_i| + O(h^3), w_0 = 0. \tag{20}$$

Increasing the index i from $i = 0$ to $i = N - 1$ conjugated with (20), produces the recursion result in the inequality

$$|w_N| \le \tfrac{\exp(hMN)-1}{hM} O(h^3) = \tfrac{\exp(hMN)-1}{M} O(h^2). \tag{21}$$

Since $\frac{\exp(hMN)-1}{M}$ is completely independent of h, the global discretization error is bounded as

$$e_N (= q_N - \hat{q}(x_N)) = hu(N\frac{\Delta}{2}, x_N) + O(h^2). \tag{22}$$

If the procedure described above is applied recursively, the asymptotic expansion can be obtained in the form

$$\hat{q}(x_i) = q_i + hu_1(i\tfrac{\Delta}{2}, x_i) + h^2 u_2(i\tfrac{\Delta}{2}, x_i) + \dots . \tag{23}$$

where the values $u_k(i\frac{\Delta}{2}, x_i)$ for $k = 1, 2, \dots$ are independent of the step size h, and $u_k(i\frac{\Delta}{2}, x_i)$ need not be calculated to apply the extrapolation method [8].

5. EXTRAPOLATION SCHEME

The h_c-step coupling coefficients $h_c q_i$, for $i = 0, 1, \dots, N_c$ can be computed by the Schur algorithm from the known input and measured scattering data, where N_c is the number of discretized layers in a coarse step size of h_c. In the same manner the $\frac{h_c}{2^j}$-step coupling coefficients $\frac{h_c}{2^j} q_i^{<2^j>}$ for $i = 0, 1, \dots, 2^j N_c$ can be computed. Since we know the asymptotic expansion of the computed coupling functions (23), we are able to cancel the global discretization errors at points $x = 0, h_c, 2h_c, \dots N_c h_c$. Let the smallest subdivided step size be $\frac{h_c}{2^\sigma}$. In the case of $\sigma = 2$ (2-stage extrapolation) we put this extrapolation scheme in order in tabular form [7].

$y_{1,1} = q_i$		
$y_{1,2} = q_i^{<2^1>}$	$y_{2,1} = y_{1,2} + \frac{y_{1,2} - y_{1,1}}{2-1}$	
$y_{1,2^2} = q_i^{<2^2>}$	$y_{2,2} = y_{1,2^2} + \frac{y_{1,2^2} - y_{1,2}}{2-1}$	$y_{3,1} = y_{2,2} + \frac{y_{2,2} - y_{2,1}}{2^2-1}$

The first subscript of y indicates the column of the tabular form and the second subscript indicates the subdivisions. In the second and third columns the global discretization errors of order h and h^2 are cancelled. Therefore, the $y_{3,1}$ can be represented as

$$\hat{q}(x_i) = y_{3,1} + O(h_c^3) \tag{24}$$

where $y_{3,1}$ has the global disretization error order of $O(h_c^3)$.

6. NUMERICAL SIMULATION AND COMPLEXITY ANALYSIS

We now demonstrate the use of the Schur algorithm combined with an extrapolation method for reconstructing the coupling function of an inhomogeneous medium for $0 \le x \le 2.25$. In order to generate scattering data by computer simulation, we directly solve (1) with a fine discretization step size of 10^{-5} using Euler's method [7] given the exact coupling profile of $\hat{q}(x)$. Thus, the generated scattering data have a global discretization error order of $O(10^{-5})$ [7].

Fig. 1 (dots) shows q_i values reconstructed directly from the computer-generated scattering data by the Schur algorithm with $h_c = 0.1$, which is very coarse discretization compared to the length of the inhomogeneous medium. The results exhibit a large discretization error propagation. This error propagation can be compensated for using the proposed method where we apply a 3-stage extrapolation. The results shown in Fig.1 (x symbols) demonstrate that an excellent reconstruction is obtained using the proposed method. Fig.2 shows the relative errors of the Schur algorithm (dots) and the proposed method (x symbols) computed by $\log_{10}\frac{|\hat{q}(x_i)-q_i|}{|\hat{q}(x_i)|}$ and $\log_{10}\frac{|\hat{q}(x_i)-q_{i,extra}|}{|\hat{q}(x_i)|}$ respectively, where $q_{i,extra}$ denotes the approximated coupling function of the proposed method. Roughly speaking, a 3-stage extrapolation can compensate for the global discretization error to 4 significant digits (Fig.2). Therefore, the approximated coupling function can be expressed as $\hat{q}(x_i) = q_{i,extra} + O(h_c^4)$, just as we expected, which means that $\log_{10}\frac{|\hat{q}(x_i)-q_{i,extra}|}{|\hat{q}(x_i)|} \le -4$. In order to reveal approximately similar accuracy of the proposed method using the Schur algorithm, the medium should be discretized with a step size of $h_c/1000 = 10^{-4}$ so that the reconstruction has a discretization error order of $O(10^{-4})$. The + symbols in Fig.2 represents the relative error of the Schur algorithm with a step size of $h_c/1000$, though they are less inaccurate than the proposed method. This relative error is computed using $\log_{10} \frac{|\hat{q}(x_i)-q_{i,1000}|}{|\hat{q}(x_i)|}$, where $q_{i,1000}$ indicates the coupling function reconstructed by the Schur algorithm with a step size of $h_c/1000$.

As for computational complexity, we neglect the addition and the shifting(delay, advance) operations, which are performed quickly compared to multiplications. For a Schur algorithm with N_c layers, the total number of multiplications equals $2N_c^2 + 2^2 N_c$ [2]. For a 3-stage extrapolation, the complexities for calculating the $\frac{h_c}{2}$, $\frac{h_c}{2^2}$, and $\frac{h_c}{2^3}$-step coupling functions are $2^3N_c^2+2^3N_c$, $2^5N_c^2+2^4N_c$ and $2^7N_c^2+2^5N_c$ respectively. $N_c \times 3!$ additional multiplications are required for a 3-stage extrapolation. Therefore, the total complexity equals $170N_c^2 + 66N_c$. In the case of the + symbols in Fig.2, the computational complexity equals $2 \cdot 10^6 N_c^2 + 2^2 \cdot 10^3 N_c$. Compared to the proposed method, $2 \cdot 10^6 N_c^2 + 2^2 \cdot 10^3 N_c$ is a notably large computational cost. In general, the Schur algorithm combined with an σ-stage extrapolation method requires $\frac{2}{3}(2^{2(\sigma+1)} - 1)N_c^2 + (2^2(2^{\sigma+1} - 1) + \sigma!)N_c$ multiplications. For the general Schur algorithm, which results in an similar accuracy as a σ-stage extrapolation, the complexity is $2\left(\frac{1}{h_c}\right)^{2\sigma} N_c^2 + 2^2\left(\frac{1}{h_c}\right)^{\sigma} N_c$. We recognize that the coefficients of the N_c^2 and N_c degree terms in complexity of the general Schur algorithm are dependent on σ and $\frac{1}{h_c}$, where, in general, the value of $\frac{1}{h_c}$ is large one. If we increase the extrapolation stage to get more accurate result, the complexity of the general Schur algorithm, which results in an similar accuracy as the proposed method, is dramatically increased. Therefore, the proposed method is more efficient in complexity than general Schur algorithm.

A high degree of accuracy with a lower computational cost is obtained for the Schur algorithm combined with an extrapolation method, as demonstrated in this simulation study.

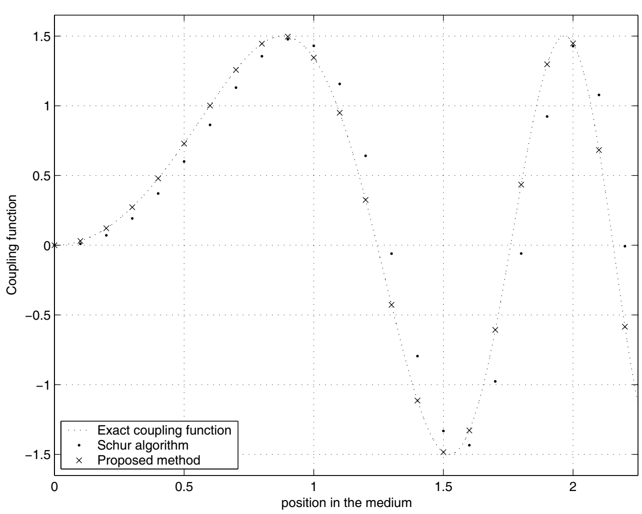

Fig. 1. Reconstruction results for Schur algorithm and proposed method

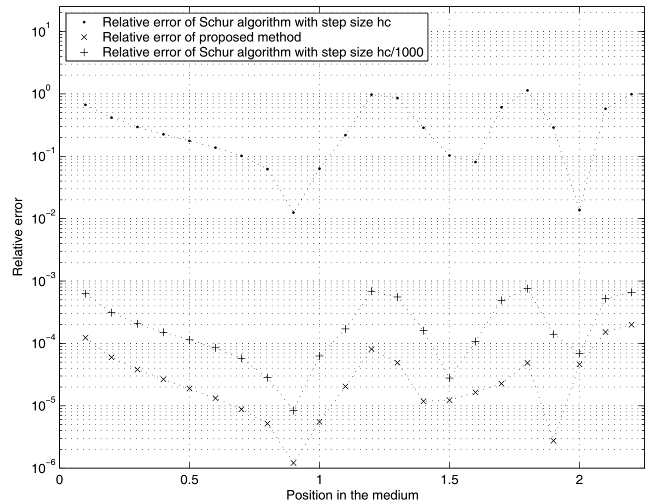

Fig. 2. Relative error plot for Schur algorithm with step size h_c, proposed method, and Schur algorithm with step size $h_c/1000$.

7. REFERENCES

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