2D ARMA PARAMETER IDENTIFICATION USING A HYBRID LATTICE DESIGN

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ABSTRACT

A new hybrid lattice structure to identify the parameters of an unknown 2D ARMA (*M*, *N*) system, where *M* and *N* can take arbitrary values different from each other, has been presented. The proposed hybrid analysis model incorporates both two-channel and single channel lattice stages in an interleaved manner. The two-channel lattice part is based on the formerly proposed 2D ARMA lattice modeling approach where only the case *M*=*N* was covered. A new formulation to calculate the ARMA parameters has been derived taking the estimated parameter b_0 into consideration.

1. INTRODUCTION

The fundamental problem of identifying a 2D (two-dimensional) system from measurements of its output response to known input excitation such as a white noise source, is one that impacts on many important fields of interest. Because of its numerical robustness and consistency, linear identification under lattice form is of special interest. Although many valid 2D AR (autoregressive) lattice structures have been developed exhibiting different properties [1-3], 2D ARMA (autoregressive moving average) lattice modeling and identification studies are not mature enough. Kayran has proposed a method [4] to obtain a 2D ARMA (*M*, *N*) lattice model for *M*=*N* using a two-channel AR lattice filter where (*M*, *N*) represents the number of points in the prediction support region corresponding to the AR and MA orders, respectively.

This paper proposes a new hybrid lattice structure for 2D ARMA (*M*, *N*) modeling and identification where *M* and *N* can take arbitrary values different from each other. The 2D ARMA system to be identified is defined as follows:

$$y(k_1,k_2) = b_0 x(k_1,k_2) + \sum_{n=1}^{N} b_n x((k_1,k_2)-n) - \sum_{m=1}^{M} a_m y((k_1,k_2)-m) \tag{1}$$

Here the notation $y((k_1, k_2)-m)$ and $x((k_1, k_2)-n)$ denotes the *m*th or *n*th element behind $y(k_1, k_2)$ or $x(k_1, k_2)$ in the prediction region. ARMA system parameters a_m , b_n and b_0 are identified provided that the system input $x(k_1, k_2)$ and the system output $y(k_1, k_2)$ are known.

2. HYBRID LATTICE DESIGN

The proposed hybrid lattice structure incorporates both two-channel and single-channel AR lattice stages in an interleaved manner. In order to compose the analysis model for an ARMA (*M*,*N*) system where $M \neq N$, we need to define *S* as the smaller, and *G* as the greater of the (*M*, *N*) pair. The compact form equations defining the two-channel lattice stages are given below. Here, *r*=1,2,..,*S* and *p*=*r*,…,*S* for each *r*.

$$\mathbf{f}_{p-r}^{(r)}(k_1,k_2) = \mathbf{f}_{p-r}^{(r-1)}(k_1,k_2) + \mathbf{K}_{p-r}^{(r)^T}\mathbf{b}_p^{(r-1)}(k_1,k_2) \tag{2.a}$$

$$\mathbf{b}_p^{(r)}(k_1,k_2) = \mathbf{b}_p^{(r-1)}(k_1,k_2) + \overline{\mathbf{K}}_p^{(r)^T}\mathbf{f}_{p-r}^{(r-1)}(k_1,k_2) \tag{2.b}$$

Here $\mathbf{f}_{p-r}^{(r)}(k_1,k_2) = [f_{u_{p-r}}^{(r)}(k_1,k_2) \quad f_{t_{p-r}}^{(r)}(k_1,k_2)]^T$ is a vector of *r*th lattice stage forward prediction errors of the first and second channels, $f_{u_{p-r}}^{(r)}(k_1,k_2)$ and $f_{t_{p-r}}^{(r)}(k_1,k_2)$, respectively. In the same manner, $\mathbf{b}_p^{(r)}(k_1,k_2) = [b_{u_p}^{(r)}(k_1,k_2) \quad b_{t_p}^{(r)}(k_1,k_2)]^T$ is a vector of *r*th lattice stage backward prediction errors of the first and second channels, $b_{u_p}^{(r)}(k_1,k_2)$ and $b_{t_p}^{(r)}(k_1,k_2)$, respectively. $\mathbf{K}_{p-r}^{(r)}$ and $\overline{\mathbf{K}}_p^{(r)}$ are the real valued 2x2 matrices of the *r*th lattice stage forward and backward reflection coefficients, respectively. They are obtained by differentiating forward and backward prediction error vectors with respect to $\mathbf{K}_{p-r}^{(r)}$ and $\overline{\mathbf{K}}_p^{(r)}$ and equating the result to zero [4].

The zeroth lattice stage forward and backward prediction error vectors of the two-channel and single channel lattice stages are defined as follows. For the two-channel lattice stages given in (3.a), *p*=1,…,*S* and for the single channel lattice stages given in (3.b), *p*=*S*+1,…,*G*.

$$\mathbf{f}_p^{(0)}(k_1,k_2) = \mathbf{b}_p^{(0)}(k_1,k_2) = \left[u((k_1,k_2)-p) \quad t((k_1,k_2)-p)\right]^T \tag{3.a}$$

$$f_p^{(0)}(k_1,k_2) = b_p^{(0)}(k_1,k_2) = \begin{cases} x(k_1,k_2) & M < N \\ y(k_1,k_2) & M \geq N \end{cases} \tag{3.b}$$

The initial channel inputs of the two-channel lattice stages are defined as follows. The first channel input $u(k_1,k_2)$ has been chosen to be the difference of $y(k_1,k_2)$ and $x(k_1,k_2)$ in order to use $x(k_1,k_2)$ in the estimation of $y(k_1,k_2)$.

$$u(k_1,k_2) \triangleq y(k_1,k_2) - x(k_1,k_2) \tag{4}$$

$$t(k_1,k_2) \triangleq \begin{cases} x(k_1,k_2) & M \geq N \\ y(k_1,k_2) & M < N \end{cases} \tag{5}$$

Single-channel AR lattice stages are treated in two parts. In the first part, the single channel lattices are interleaved with two-channel ones and *r*=1,…,*S* and *p*=*S*+1,…,*G*. In the second part there are only single channel lattices for r=S+1,..,G and p=r,…,G. The compact form equations defining the single-channel lattice stages are given below. $k_{f_{p-r}}^{(r)}$ and $k_{b_p}^{(r)}$ are the forward and backward reflection coefficients, respectively [1].

$$\begin{bmatrix} f_{u_{p-r}}^{(r)}(k_1,k_2) \\ b_{u_p}^{(r)}(k_1,k_2) \end{bmatrix} = \begin{bmatrix} 1 & k_{b_p}^{(r)} \\ k_{f_{p-r}}^{(r)} & 1 \end{bmatrix} \begin{bmatrix} f_{u_{p-r}}^{(r-1)}(k_1,k_2) \\ b_{u_p}^{(r-1)}(k_1,k_2) \end{bmatrix} \tag{6}$$

The b_0 parameter is not readily available from the calculated lattice parameters, since all lattice stages contain AR recursions. We have modified the b_0 estimate given in [4] in accordance with our channel inputs and obtained the following formulas.

$$\hat{b}_0 = 1 + \frac{E[f_{u_0}^M(k_1,k_2) \;\; f_{t_0}^M(k_1,k_2)]}{E[f_{t_0}^M(k_1,k_2)^2]} \quad \text{for} \quad M \geq N \tag{7.a}$$

$$\hat{b}_0 = \frac{E[(f_{t_0}^M(k_1,k_2)(f_{y_0}^M(k_1,k_2) \; - \; f_{u_0}^M(k_1,k_2))]}{E[(f_{t_0}^M(k_1,k_2) \; - \; f_{u_0}^M(k_1,k_2))^2]} \quad \text{for} \quad M < N \tag{7.b}$$

3. CALCULATION OF THE ARMA PARAMETERS

We propose a new method to calculate the ARMA parameter vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, once we have obtained the lattice reflection coefficients, the tap weights of the forward prediction error filters of both channels and the estimated parameter $\hat{b}_0$. Here $\hat{\mathbf{a}}$ represents the identified AR parameters and $\hat{\mathbf{b}}$ represents the identified MA parameters of the ARMA model as defined below.

$$\hat{\mathbf{a}} \triangleq [\hat{a}_1 \;\; \hat{a}_2 \;\; \ldots \;\; \hat{a}_M]^T \quad \hat{\mathbf{b}} \triangleq [\hat{b}_1 \;\; \hat{b}_2 \;\; \ldots \;\; \hat{b}_N]^T \tag{8}$$

In order to calculate the forward and backward prediction error filter tap weights, we use the single channel and two-channel Levinson-Durbin recursions; the latter given below for convenience.

$$\mathbf{a}_{p-r}^{(r)} = \begin{bmatrix} \mathbf{a}_{u_{p-r}}^{(r)} \\ \mathbf{a}_{t_{p-r}}^{(r)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_{u_{p-r}}^{(r)} \\ \hat{\mathbf{a}}_{t_{p-r}}^{(r)} \end{bmatrix} - \mathbf{K}_{p-r}^{(r)T} \begin{bmatrix} \hat{\mathbf{g}}_{u_p}^{(r)} \\ \hat{\mathbf{g}}_{t_p}^{(r)} \end{bmatrix} \tag{9.a}$$

$$\mathbf{g}_{p}^{(r)} = \begin{bmatrix} \mathbf{g}_{u_p}^{(r)} \\ \mathbf{g}_{t_p}^{(r)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{g}}_{u_p}^{(r)} \\ \hat{\mathbf{g}}_{t_p}^{(r)} \end{bmatrix} - \bar{\mathbf{K}}_{p}^{(r)^T} \begin{bmatrix} \hat{\mathbf{a}}_{u_{p-r}}^{(r)} \\ \hat{\mathbf{a}}_{t_{p-r}}^{(r)} \end{bmatrix} \tag{9.b}$$

Here $\mathbf{a}_{u_{p-r}}^{(r)}$ and $\mathbf{a}_{t_{p-r}}^{(r)}$ denote the rth order tap weight coefficient vectors of the forward prediction error filters related to the forward prediction of the (p-r)th point for the first and second channels, respectively. $\mathbf{g}_{u_p}^{(r)}$ and $\mathbf{g}_{t_p}^{(r)}$ denote the rth order tap weight coefficient vectors of the backward prediction error filters related to the backward prediction of the pth point for the first and second channels, respectively. For the case M=N, we have derived the following equations for the ARMA parameter estimates the proofs of which are given in Appendix.

$$\hat{\mathbf{a}} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{u_0}^{(M)}(1) \\ \vdots \\ \mathbf{a}_{u_0}^{(M)}(M) \end{bmatrix} + (1-\hat{b}_0) \begin{bmatrix} \mathbf{a}_{t_0}^{(N)}(1) \\ \vdots \\ \mathbf{a}_{t_0}^{(M)}(M) \end{bmatrix} \tag{10.a}$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_N \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{u_0}^{(M)}(M+1) \\ \vdots \\ \mathbf{a}_{u_0}^{(M)}(M+N) \end{bmatrix} + (1-\hat{b}_0) \begin{bmatrix} \mathbf{a}_{t_0}^{(N)}(M+1) \\ \vdots \\ \mathbf{a}_{t_0}^{(M)}(N+M) \end{bmatrix} \tag{10.b}$$

For the case $M > N$, the following matrix form equations hold for the ARMA parameter estimates.

$$\hat{\mathbf{a}} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_N \\ \hat{a}_{N+1} \\ \vdots \\ \hat{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{u_0}^{(M)}(1) + (1-\hat{b}_0)\mathbf{a}_{t_0}^{(N)}(1) \\ \vdots \\ \mathbf{a}_{u_0}^{(M)}(N) + (1-\hat{b}_0)\mathbf{a}_{t_0}^{(N)}(N) \\ \mathbf{a}_{u_0}^{(M)}(N+1) \\ \vdots \\ \mathbf{a}_{u_0}^{(M)}(M) \end{bmatrix} \tag{11.a}$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_N \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{u_0}^{(M)}(M+1) + (1-\hat{b}_0)\mathbf{a}_{t_0}^{(N)}(M+1) \\ \vdots \\ \mathbf{a}_{u_0}^{(M)}(M+N) + (1-\hat{b}_0)\mathbf{a}_{t_0}^{(M)}(N+M) \end{bmatrix} \tag{11.b}$$

For the case $M < N$, the following matrix form equations hold for the ARMA parameter estimates the proofs of which are given in Appendix.

$$\hat{\mathbf{a}} = \begin{bmatrix} \mathbf{a}_{u_0}^{(N)}(1) + (1-\hat{b}_0)\left(\mathbf{a}_{t_0}^{(M)}(1) - \mathbf{a}_{u_0}^{(M)}(1)\right) \\ \vdots \\ \mathbf{a}_{u_0}^{(N)}(M) + (1-\hat{b}_0)\left(\mathbf{a}_{t_0}^{(M)}(M) - \mathbf{a}_{u_0}^{(M)}(M)\right) \end{bmatrix} \tag{12.a}$$

$$\hat{\mathbf{b}} = \begin{bmatrix} -\mathbf{a}_{u_0}^{(N)}(M+1) + (1-\hat{b}_0)(\mathbf{a}_{u_0}^{(M)}(M+1) - \mathbf{a}_{t_0}^{(M)}(M+1)) \\ \vdots \\ -\mathbf{a}_{u_0}^{N}(2M) + (1-\hat{b}_0)(\mathbf{a}_{u_0}^{(M)}(2M) - \mathbf{a}_{t_0}^{(M)}(M+N)) \\ -\mathbf{a}_{u_0}^{(N)}(2M+1) \\ \vdots \\ -\mathbf{a}_{u_0}^{(N)}(M+N) \end{bmatrix} \tag{12b}$$

The hybrid lattice design algorithm described above has been summarized for M≠N in Algorithm 1 in 11 steps. It should be noted that for the case M=N, the whole structure reduces to two-channel lattice stages, hence there is no hybrid structure involved.

4. EXPERIMENTAL RESULTS

To illustrate the functionality and effectiveness of the proposed 2D hybrid lattice design, we give here a computer simulation example where AR order is smaller than the MA order (M=3 and N=8). The problem we wish to solve is simultaneously identifying the AR and MA parameters of the unknown 2D ARMA (3,8) system given the data fields $x(k_1, k_2)$ and $y(k_1, k_2)$. The original ARMA parameters have been chosen so as to satisfy the stability conditions [1]. The system input $x(k_1, k_2)$ is white-Gaussian noise with variance σ_x^2 and mean zero.

Algorithm 1. Hybrid Lattice Algorithm for $M \neq N$.

Step 1:Determine *S* and *G*.
$M > N \Rightarrow S=N$ and $G=M$
$N > M \Rightarrow S=M$ and $G=N$
Step 2: Determine the hybrid structure.
Two-channel: $r=1,\ldots,S$ and $p=r,\ldots,S$ $\forall r$
Single channel: $r=1,\ldots,S$ and $p=S+1,\ldots,G$ $\forall r$
$r=S+1,\ldots,G$ and $p=r,\ldots,G$ $\forall r$
Step 3: Initialization: $r=0$
$p=1,\ldots S$ use Eq. (3.a)
$p=S+1,\ldots,G$ use Eq. (3.b).
Step 4: $r=r+1$. $p=r$. If $r<G$ go to Step 5 else go to Step 10.
Step 5: See Step 2 if two-channel or single channel.
Step 6: Calculate Reflection Coefficients
Calculate $\mathbf{K}_{p-r}^{(r)}$ and $\overline{\mathbf{K}}_{p}^{(r)}$ for two-channel.
Calculate $k_{f_{p-r}}^{(r)}$ and $k_{b_p}^{(r)}$ for single channel.
Step 7: Forward and Backward Prediction Errors
Calculate using Eq. (2.a) and (2.b) for two-channel.
Calculate using Eq. (6) for single-channel.
Step 8: Calculate Tap-weight Coefficients
Calculate using Eq. (9.a) and (9.b) for two-channel.
Calculate using Levinson-Durbin for single-channel.
Step 9: $p=p+1$. If $p < G$ go to Step 5 else go to Step 4.
Step 10: Calculate $\hat{b}_0$ using Eq (7).
Step 11: Obtain ARMA estimates using Eq. (11) or (12).

We have compared the estimates obtained using our new hybrid lattice algorithm with the LS (Least Squares) estimates. The identification results are listed in Table 1 for different data field sizes, the first row showing the hybrid lattice and the second row showing the LS estimates. Both the lattice and LS estimates have been calculated assuming that the ordering of the data points in the 2D prediction region is known.

The performance of the developed method has been evaluated by the Itakura-Saito distance measure [5], L_1, L_2 and L_∞ vector norms. By the use of Itakura-Saito distance, a measure for the similarity between original and estimated power spectrums has been obtained. All the performance criteria have been listed in Table 2 for both the lattice and LS estimates.

By looking at the Itakura-Saito distances, we can infer that there is a close match between the power spectrums obtained using original and identified parameters by both lattice and LS algorithms. Lattice algorithm gives a better power spectrum match (a lower Itakura-Saito distance) for small data sizes. The L_1, L_2 and L_∞ norms show that obtained parameter estimate vectors are quite close to the original ones.

5. CONCLUSIONS

We proposed a new hybrid lattice algorithm for identifying the parameters of an unknown 2D ARMA (M,N) system. The main advantage of the proposed algorithm is that M and N can take arbitrary values different from each other. We have also derived new formulas for the ARMA parameter estimates taking estimated parameter b_0 into consideration, thus resulting in a more reliable parameter identification. We have compared the results with the LS estimates and have seen that our hybrid lattice identification results closely converges to the LS estimates for long data records, thus verifying this newly proposed structure. For short data records, the results of the hybrid lattice algorithm prove to be better than the LS estimates. This hybrid lattice algorithm inherits the modular structure of the conventional lattices provided that the ordering of the 2D data points is appropriately chosen. One major advantage compared to LS is that there is no need for matrix inverse operations.

Table 1. Identification results for the Lattice and LS algorithms.

Original ARMA Parameters		Identified ARMA Parameters LAT and LS		
		Data Field Size		
		10x10	**50x50**	**100x100**
AR Parameters	0.3	0.3878 0.3942	0.3063 0.2986	0.3154 0.3051
	-0.3	-0.0913 -0.1303	-0.2184 -0.2431	-0.2251 -0.2751
	0.2	0.4097 0.3274	0.2893 0.2766	0.2830 0.2297
MA Parameters	0.2	0.1929 0.2355	0.2010 0.2062	0.1994 0.2058
	0.2	0.4131 0.3985	0.2310 0.2435	0.2258 0.2195
	0.2	0.2981 0.2996	0.4315 0.2627	0.3745 0.2233
	-0.2	-0.1681 -0.2054	-0.2038 -0.1975	-0.1932 -0.1986
	0.2	0.3399 0.2431	0.2430 0.2352	0.2384 0.2142
	-0.3	-0.1827 -0.1874	-0.2357 -0.2556	-0.2403 -0.2808
	0.3	0.1952 0.2655	0.2496 0.2587	0.2592 0.2868
	-0.5	-0.4853 -0.5414	-0.4967 -0.5029	-0.5047 -0.5002
b_0	0.8	0.8281 0.7402	0.8585 0.7949	0.8580 0.7955

Table 2. Performance criteria for the Lattice and LS algorithms.

Performance Criteria LAT and LS	Data Field Size		
	10x10	**50x50**	**100x100**
L_1-norm	1.2613 1.0218	0.6639 0.3787	0.5827 0.1609
L_2-norm	0.4435 0.3543	0.2849 0.1412	0.2324 0.0570
L_∞-norm	0.2131 0.1985	0.2315 0.0766	0.1745 0.0297
Itakura-Saito Distance	0.3641 0.4475	0.2733 0.0489	0.2710 0.0047

6. REFERENCES

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APPENDIX

The first and second channel prediction error fields of the last prediction orders for the case $M = N$ can be defined as follows .

$$f_{u0}^{M}(k_1,k_2) \triangleq y(k_1,k_2) - x(k_1,k_2) - \sum_{m=1}^{M} \alpha_m y((k_1,k_2)-m) - \sum_{n=1}^{N} (\beta_n - \alpha_n) x((k_1,k_2)-n) \tag{A.1}$$

$$f_{x0}^{N}(k_1,k_2) \triangleq x(k_1,k_2) - \sum_{n=1}^{N} (\gamma_n - \theta_n) x((k_1,k_2)-n) - \sum_{m=1}^{M} \theta_m y((k_1,k_2)-m) \tag{A.2}$$

If we substitute $x(k_1, k_2)$ derived from (A.2) in (1) and (A.1) respectively, we obtain the following equations for $y(k_1, k_2)$.

$$y(k_1,k_2) = b_0 f_{x0}^{N}(k_1,k_2) + \sum_{m=1}^{M} (b_0\theta_m - a_m) y((k_1,k_2)-m) + \sum_{n=1}^{N} (b_0(\gamma_n - \theta_n) + b_n) x((k_1,k_2)-n) \tag{A.3}$$

$$\hat{y}(k_1,k_2) = f_{u0}^{M}(k_1,k_2) + f_{x0}^{N}(k_1,k_2) + \sum_{m=1}^{M} (\theta_m + \alpha_m) y((k_1,k_2)-m) + \sum_{n=1}^{N} (\gamma_n - \theta_n + \beta_n - \alpha_n) x((k_1,k_2)-n) \tag{A.4}$$

We now compare (A.3) and (A.4) to conclude with the following relations for the AR and MA coefficients of the ARMA model.

$$\hat{a}_m = -\alpha_m - (1-b_0)\theta_m \qquad m = 1\ldots M \tag{A.5}$$

$$\hat{b}_n = (\beta_n - \alpha_n) + (1-b_0)(\gamma_n - \theta_n) \qquad n = 1\ldots N \tag{A.6}$$

$-\alpha_m$ is the $\mathbf{a}_{u_0}^{(M)}(m)$ and $-\theta_m$ is the $\mathbf{a}_{t_0}^{(N)}(m)$ given in (10.a), $m=1,\ldots,M$. Likewise $(\beta_n-\alpha_n)$ is the $\mathbf{a}_{u_0}^{(M)}(n)$ and $(\gamma_n-\theta_n)$ is the $\mathbf{a}_{t_0}^{(N)}(n)$ given in (10.b), $n=M+1,\ldots,M+N$.

The Mth order first and second channel prediction error fields for the case $M > N$ can be defined as follows.

$$f_{u0}^{M}(k_1,k_2) \triangleq y(k_1,k_2) - x(k_1,k_2) + \sum_{m=1}^{M} \alpha'_m x((k_1,k_2)-m) - \sum_{m=1}^{M} (\alpha'_m + \beta'_m) y((k_1,k_2)-m) \tag{A.7}$$

$$f_{y0}^{M}(k_1,k_2) \triangleq y(k_1,k_2) - \sum_{m=1}^{M} (\gamma'_m + \theta'_m) y((k_1,k_2)-m) + \sum_{m=1}^{M} \theta'_m x((k_1,k_2)-m) \tag{A.8}$$

The final Nth ($r = N$; $p = N$) order forward prediction error field of the first channel can be defined as follows:

$$f_{y0}^{N}(k_1,k_2) \triangleq y(k_1,k_2) - \sum_{m=1}^{M} \alpha_m y((k_1,k_2)-m) - x(k_1,k_2) - \sum_{n=1}^{N} \beta_n x((k_1,k_2)-n) \tag{A.9}$$

If we substitute $x(k_1, k_2)$ derived from (A.7) in (1) and (A.9) respectively, we obtain the following equations for $y(k_1, k_2)$.

$$\hat{y}(k_1,k_2) = f_{y0}^{N}(k_1,k_2) - f_{u0}^{M}(k_1,k_2) + f_{y0}^{M}(k_1,k_2) + \sum_{m=1}^{M} (\alpha_m + ((\gamma'_m + \theta'_m) - (\alpha'_m + \beta'_m))) y((k_1,k_2)-m) + \sum_{n=1}^{M} (\beta_n - (\theta'_n - \alpha'_n)) x((k_1,k_2)-n) + \sum_{n=M+1}^{N} \beta_n x((k_1,k_2)-n) \tag{A.10}$$

$$\hat{y}(k_1,k_2) = b_0(f_{y0}^{M}(k_1,k_2) - f_{u0}^{M}(k_1,k_2)) - \sum_{m=1}^{M} (a_m - b_0((\gamma'_m + \theta'_m) - (\alpha'_m + \beta'_m))) y((k_1,k_2)-m) + \sum_{n=1}^{M} (b_n - b_0(\theta'_n - \alpha'_n)) x((k_1,k_2)-n) + \sum_{n=M+1}^{N} b_n x((k_1,k_2)-n) \tag{A.11}$$

We now compare (A.10) and (A.11) to conclude with the following relations for the AR and MA coefficients of the ARMA model.

$$\hat{a}_m = -\alpha_m + (1-b_0)((\alpha'_m + \beta'_m) - (\gamma'_m + \theta'_m)) \quad m = 1\ldots M$$

$$\hat{b}_n = \beta_n + (1-b_0)(\alpha'_n - \theta'_n) \; n=1,..,M \text{ and } \hat{b}_n = \beta_n \; n=M+1,..N \tag{A.12}$$