A Signal-processing Interpretation of the Riemann Zeta Function

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ABSTRACT

This paper will present a signal-processing model of the Riemann Zeta function (and other related functions), and show how the Riemann Hypothesis (RH) can be re-cast as a signal-processing problem. This study also yields a new signal-processing paradigm that we will call "Discrete log-time systems". These systems differ in many fundamental ways from their conventional discrete-time cousins, and their study leads one to a fascinating world at the intersection of signal-processing and number theory.

1. INTRODUCTION

In 1859, the mathematician Riemann wrote a startling paper that showed that the distribution of prime numbers could be exactly predicted by a formula that gives the "average" distribution of primes, plus a term consisting of an infinite summation of sinusoids that represent the exact fluctuation around this average. The frequency of these sinusoidal waves corresponded to the imaginary part of the "non-trivial" zeros of a complex function known as the eta function $\eta(s)$, where the term "non-trivial" refers to those zeros that lie on the line RE(s) = +1/2. Riemann asserted that in a particular strip of the complex plane bounded by 0 < RE(s) < 1, the ONLY zeros that are present lie on the vertical line RE(s) = ½. This assertion is of fundamental importance in many disparate fields, and to this day it remains one of the most famous unproven theorems of the last 100 years.

2. THE ZETA FUNCTION $\varsigma(s)$

The function $\varsigma(s)$ is a complex function of the complex variable s given by the following formula [1];

1) $\varsigma(s) = \sum_{k=1}^{\infty} k^{-s} = 1 + 2^{-s} + 3^{-s} \ldots\ldots$

where s is a complex variable.
This series converges for RE(s) > 1, and therefore cannot be directly evaluated on the "interesting" vertical line RE(s) = ½. Zeta(s) has a simple pole at s=1.

The famous mathematician Euler showed that zeta(s) could also be written in product form as;

2) $\varsigma(s) = \sum_{k=1}^{\infty} k^{-s} = \prod_{p} \frac{1}{1 - p^{-s}}$,

where the product term is taken overall all primes P. This formula provides a link between the zeta function and prime numbers, and will be derived later using simple signal-processing flow-graph manipulations.

3. THE ETA FUNCTION $\eta(s)$

The eta function $\eta(s)$ is also known as the "alternating zeta function", and is given by;

3) $\eta(s) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-s} = 1 - 2^{-s} + 3^{-s} - 4^{-s} \ldots\ldots\ldots$

$\eta(s)$ is similar to $\varsigma(s)$ except that the terms alternate in sign. This modification results in an extension of the region of convergence. $\eta(s)$ and $\varsigma(s)$ are related by the following [2] ;

4) $\varsigma(s) = \eta(s)/(1 - 2 \cdot 2^{-s})$

The term on the right converges for RE(s) > 0, and therefore $\eta(s)$ can be used in place of $\varsigma(s)$ to search for zeros along the critical line RE(s) = ½, since both have the same zeros on the critical line.

4. LINK TO SIGNAL-PROCESSING NETWORKS

The work in this paper was initially motivated by the simple desire to search for zeros of $\eta(s)$ on the line RE(s) = 1/2 using a SPICE simulator. To accomplish this, a signal-processing network must be found whose Laplace transform, when evaluated on the imaginary axis, yields a complex response that is identical to evaluating $\eta(s)$ on the line RE(s) = ½. This can be done by shifting the complex plane by -1/2, using the following variable substitution.

1) $s' = s - \dfrac{1}{2}$,

where s' represents the Laplace transform variable.

If this variable substitution is applied to the terms k^{-s}, we get (after minor manipulations);

6) $k^{-s} = k^{-1/2} \cdot e^{-s' \cdot \ln(k)}$

$\eta(s)$ can be then be written as;

7) $\eta(s) = \sum_{k=1}^{\infty} (-1)^{k-1} \cdot k^{-s} = 1 - 2^{-s} + 3^{-s} \ldots$

$$= 1 - 2^{-1/2} e^{-s' \cdot \ln(2)} + 3^{-1/2} e^{-s' * \ln(3)} \quad \ldots$$

For those schooled in signal-processing theory, the term $e^{-s' * \ln(k)}$ looks quite familiar; it is the transfer function of a signal-processing block that has a simple delay of ln(k) seconds. More specifically,

8) $L(\partial(T)) = e^{-sT}$,

where L is the Laplace Transform operator and $\partial(T)$ is a Dirac impulse that occurs at time T. Equation 7 can therefore be interpreted as the summation of weighted ideal delay units, each with a Laplace transfer function of $k^{1/2} \cdot e^{-s' * \ln(k)}$.

To meet the goal of using a SPICE simulator to evaluate the magnitude response of $\eta(s)$ along the line RE(s) = ½, it is convenient to express equation 7 as the transfer function of some network that can be easily entered into a schematic drawing program. Equation 7 can be directly interpreted as network consisting of a sum of linear weighted delays, as shown below.

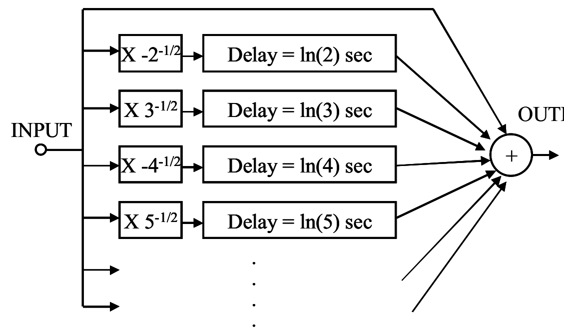

Figure 1. Signal-processing equivalent of the eta function.

An "FIR" implementation is also possible, where a single tapped delay line replaces the delay blocks of fig. 1.
The following figure shows a Spice frequency-response simulation of the network of figure 1, with the first 90 delay terms entered as schematic blocks. The frequency is swept from 1 to 8.5 Hz, revealing the low-frequency zeros of the eta function. The frequencies of these notches are exactly aligned to the published frequencies of eta(s) zeros. As more delay terms are added, the notches become deeper, as expected.

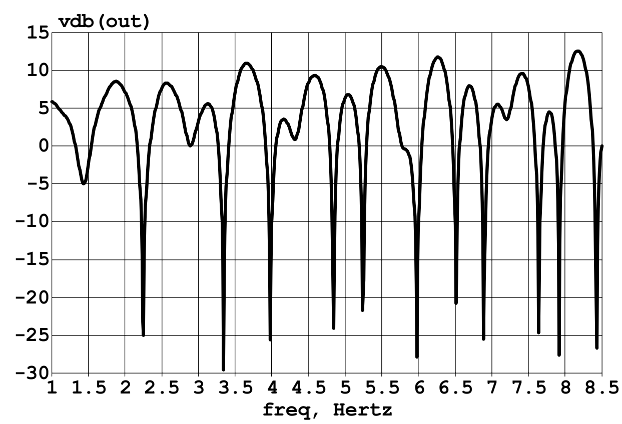

Figure 2. ***Spice frequency sweep of circuit with 90 delay terms.***

5. DISCRETE LOG-TIME (DLT) SYSTEMS

We will refer to linear networks that only contain weighted ideal delay units with delays that fall on an ln(k) time grid as discrete log-time (DLT) systems.

To simplify future notation, we introduce an operator called wld(n) (for Weighted Logarithmic Delay);

9) $wld(n) = n^{-1/2} e^{-s' * \ln(n)}$

This equation represents a term that introduces a constant time-domain delay of ln(n) seconds with an amplitude weighting of $n^{-1/2}$.

Using this new notation, equation 7 can be re-written;

10) $\varsigma(s') = 1 - wld(2) + wld(3) - wld(4)\cdots$

A very important identity involving eq 9 can be derived as follows;

11)

$$wld(n) \cdot wld(m) = n^{-1/2} \cdot m^{-1/2} \cdot e^{-s' \cdot \ln(n)} \cdot e^{-s' \cdot \ln(m)} = wld(n \cdot m)$$

This equation can be interpreted to mean that the series connection of two delay units, one with a delay of ln(n) and gain=1/sqrt(n) and the other with a delay of ln(m) and gain= 1/sqrt(m) is identical to a single delay unit with delay ln(n*m) and gain (1/sqrt(n*m)).

6. PROPERTIES OF DLT SYSTEMS

The signal-processing systems shown previously are unique in that they all have impulse responses that are non-zero ONLY at times equal to t = ln(K), K = 1 to infinity. For example, the impulse response of the eta(s) flowgraph shown in figure 1 can be determined by inspection, and is plotted below.

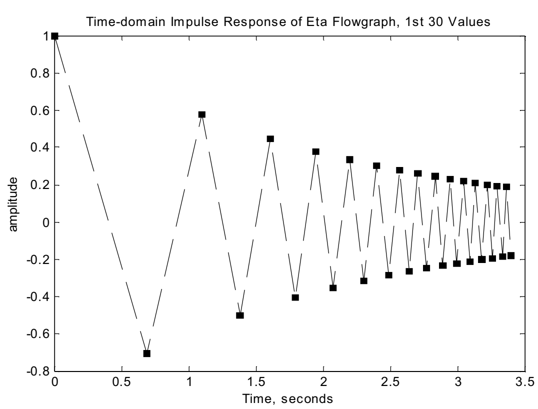

Figure 3. Impulse response of the eta network.

Systems that fall in this category have some very unique properties, listed below.

- Series-Connection Property. Two or more DLT systems placed in series produce an impulse response that is also DLT.
- Time-Shift Property. If a DLT impulse response is shifted in time by ln(kshift) seconds, then the impulses of the resulting sequence fall on a time grid defined by ln(k*kshift), k any integer. Therefore, the shifted sequence is aligned with the original sequence once every kshift samples.
- Convolution property. The impulse response of two DLT systems in series is the convolution of their individual impulse responses. At a particular time t=ln(k), the output is the sum of all paths through the combined network that have a net delay equal to ln(k) seconds. This only occurs if the individual components of a particular path have wld indexes whose product equals k (see eq. 11). One consequence of this reasoning is that outputs at time t=ln(P), P prime, can only have paths that include a single delay element, since P cannot be factored.

It is instructive to compute the average power of the eta impulse response of fig. 3. Surprisingly, the average power converges to a constant. The energy lost in the decaying amplitude envelope is exactly counter-balanced by the increasing density of impulses as time progresses.

7. EULER PRODUCT SIGNAL-PROCESSING NETWORK

Euler's famous theorem links the zeta function to prime numbers using the product formula below.

12) $\varsigma(s) = \sum_{k=1}^{\infty} k^{-s} = \prod_{p} \frac{1}{1 - p^{-s}}$, product taken over all primes p.

Using the WLD notation, we can again make a signal-processing equivalent that has the same frequency response as eq. 12evaluated on the line RE(s) = ½.

13) $H(s') = \prod_{p} \frac{1}{1 - wld(p)}$

The product operation above implies a series connection of signal-processing units, shown below;

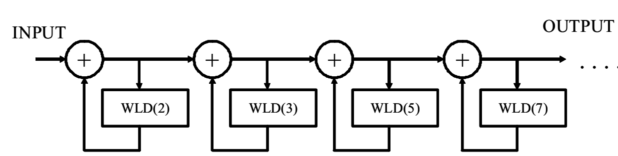

Figure 4. Signal-processing equivalent to the Euler Product-form of the zeta function.

Note that the delay units utilize ONLY the ln(prime) delay terms, and no others. This leads to an interesting property of this network; since every integer N has a set

of unique prime factors, the output of the network above has a unique path for an output impulse that occurs at t=ln(N). For example, at time t=ln(12), the ONLY path through the network is WLD(12) = WLD(2)*WLD(2)*WLD(3) resulting in the following unique impulse path;
path = twice around the WLD(2) loop and once around the WLD(3) loop. This reduction of the zeta flowgraph to a factored form provides the main link between DLT systems and prime numbers.

8. THE ZETA OSCILLATOR

The Euler-zeta network of Figure 4 may be inverted, causing all the zeroes on the imaginary axis to become poles on the imaginary axis. The inversion of Figure 4 may be accomplished trivially using standard flow-graph manipulation, resulting in the flow-diagram below.

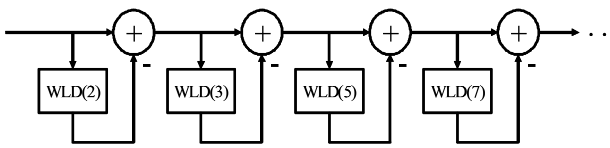

Figure 5. Inverted Euler-Zeta flow diagram.

A path analysis of figure 5 yields some surprising results. The output at time ln(n) for an impulsive input is;

0, if n has any repeated factors;
$n^{-1/2}$, if n contains an even number of unique prime factors;
$-n^{-1/2}$, if n contains an odd number of unique prime factors.

The figure below shows the first 30 values of the impulse response of this network.

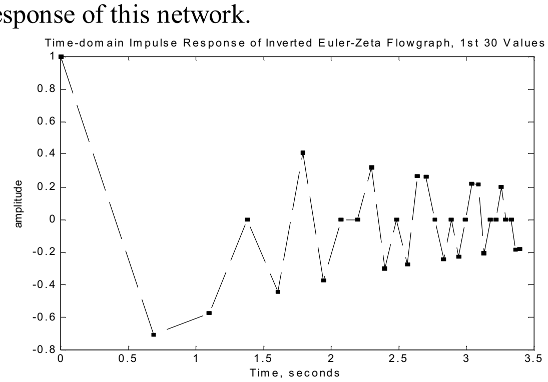

Figure 6. Time-domain Response of Inverted Euler-Zeta Flowgraph, 1st 30 Values

The inversion of the Zeta function causes zeros to become poles. The following figure shows the resulting of taking the DFT of the first 50000 time-domain samples of the impulse response. The spectral peaks are exactly aligned to known zero frequencies of the eta function.

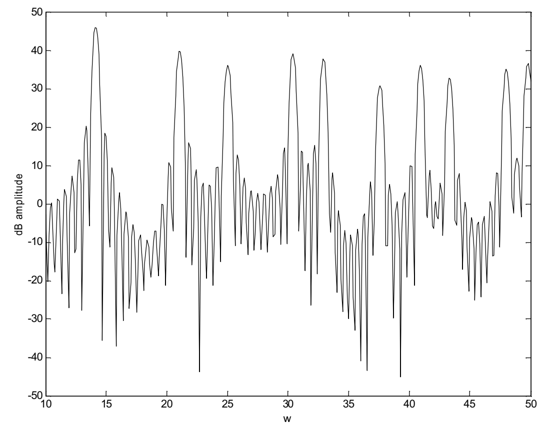

Figure 7. Frequency response of fig. 5.

The impulse response of the system shown in fig. 5 can be proven to have constant average power over time. This fact suggests that the Riemann Hypothesis is true, based on the following reasoning. It has been proven that if the Zeta zeros are not on the line RE(s) = ½, then they must exist in pairs mirrored around this line; that is, if a zero is found at s = (0.5 + epsilon) + jw1, then another zero must exist at s = (0.5 – epsilon) + jw1. If any zeros were not on the critical line, then by shifting the complex plane by -1/2 and inverting the resulting system, a pair of poles would be formed, one slightly to the left of the imaginary axis and one slightly to the right. It follows that if the original zeros were not on the critical line, the inverted Zeta-oscillator system would have an impulse response with increasing power over time.

9. CONCLUSION

This paper has introduced a novel signal-processing paradigm that has deep links to the Riemann Zeta function and hence to the distribution of prime numbers. It allows for easy visualization and manipulation of concepts that were previously buried in heavy mathematics.

10. REFERENCES

[1] Edwards, H. M. *"Riemann's Zeta Function"*, Academic Press, New York, 1974, pg 11.
[2] HTTP://MATHWORLD.WOLFRAM.COM/ RIEMANNZETAFUNCTION.HTML