ON ESTIMATING THE SIGNAL TO NOISE RATIO FROM BPSK SIGNALS

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ABSTRACT

Signal-to-noise ratio is an important parameter in many receivers. In this contribution, we derive a fixed-point equation whose solution coincides with the maximum-likelihood (ML) estimate of the amplitude of a binary phase-shift keying modulated signal and the variance of the additive white Gaussian noise. The resulting fixed-point equation is efficiently solved in a few iterations.

1. INTRODUCTION

Knowledge of the receiver's signal-to-noise ratio (SNR) is advantageous in wireless communication systems. The SNR is rarely known *a priori* and must be estimated instead. For example, SNR estimates are typically employed in soft decoding procedures, transmit power control, and handover.

Several authors have investigated SNR estimation algorithms for BPSK and QPSK signals, cf. [1, 2, 3, 4]. Cramér-Rao bounds for this estimation problem were derived in [5]. Most of the proposed estimators work well for high SNRs, but exhibit significant bias in the low SNR regime which is the regime of interest in mobile communications applications.

A notable exception is the contribution by Li, DiFazio, and Zeira [4] who derived an estimator with low bias in the region of low SNRs. In [4] a necessary condition was derived for the maximum-likelihood (ML) estimate and the resulting equation was solved iteratively. In this contribution, we follow the same line of attack, but the iterative algorithm which we propose and investigate is different.

First, we derive a sufficient statistic for the problem at hand. Next, we reformulate the necessary condition for the ML estimate as a fixed-point equation which links the decision-directed estimate to the ML estimate. The resulting two-by-two system of fixed-point equations is solved in a few iterations in a natural way.

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2. PROBLEM FORMULATION

We consider a binary phase shift keying signal in additive white Gaussian noise. We observe realizations $(x_1, \dots, x_n)$ of the following model

$$X_k = \mu B_k + U_k, \qquad (k = 1, 2, \dots, n) \tag{1}$$

where μ is a real-valued unknown parameter, $B_1, B_2, \dots, B_n$ are discrete i.i.d. random variables with

$$\mathrm{P}[B_k = -1] = \mathrm{P}[B_k = +1] = \frac{1}{2}$$

and $(U_1, U_2, \dots, U_n)$ is a multi-variate zero-mean Gaussian random vector with covariance matrix $\sigma^2 \boldsymbol{I}_n$. We are interested in estimating the unknown parameters μ and σ^2, and the signal to noise ratio $\gamma = \frac{\mu^2}{\sigma^2}$.

3. PARAMETER ESTIMATION

Proposition 1 *We say that X is a Bigaussian random variable, $X \sim \mathrm{BG}(\mu, \sigma^2)$ if its density is described by*

$$f_X(x) = \frac{1}{2} \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}} + \frac{1}{2} \frac{\exp\left(-\frac{(x+\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}} . \tag{2}$$

Without loss of generality, we assume $\mu \geq 0$ and $\sigma \geq 0$.

Proposition 2 *We define the first and second absolute moments of X through*[1]

$$\theta = \mathrm{E}\left[|X|\right] = \mu \operatorname{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right) + \frac{2\sigma}{\sqrt{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \tag{3}$$

$$\varphi = \mathrm{E}\left[X^2\right] = \sigma^2 + \mu^2 \tag{4}$$

The log-likelihood function $L(\mu, \sigma^2)$ of an i.i.d. sample $\mathcal{X} = (x_1, \dots, x_n)$ is

$$L(\mu, \sigma^2) = -n \log \sigma + \sum_{k=1}^{n} \log\left(\exp\left(-\frac{(x_k-\mu)^2}{2\sigma^2}\right) + \exp\left(-\frac{(x_k+\mu)^2}{2\sigma^2}\right)\right). \tag{5}$$

[1]The error function is related to the Q-function through $\operatorname{erf}(x) = 1 - 2Q(x\sqrt{2})$.

Now we use

$$\log(\mathrm{e}^{\delta_1} + \mathrm{e}^{\delta_2}) = \max(\delta_1, \delta_2) + \log(1 + \mathrm{e}^{-|\delta_1 - \delta_2|}) \quad (6)$$

and we see that the log-likelihood can be written as

$$\begin{aligned} L(\mu, \sigma^2) &= -n \log \sigma - \sum_{k=1}^{n} \frac{(|x_k| - \mu)^2}{2\sigma^2} \\ &+ \sum_{k=1}^{n} \log\left(1 + \exp\left(-\frac{2\mu|x_k|}{\sigma^2}\right)\right) \quad (7) \end{aligned}$$

Theorem 1 *Given an i.i.d. sample* $\mathcal{X} = (x_1, \ldots, x_n)$ *drawn from a Bigaussian distribution, then* $T(\mathcal{X}) = (|x_1|, \ldots, |x_n|)$ *is a sufficient statistic.*

The proof follows from (7) where it is seen that the log-likelihood function depends on the data through $|x_k|$.

Theorem 2 *The* conditional *ML estimates, conditioned on the realization* $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ *are*

$$\begin{aligned} \hat{\mu}_0(\boldsymbol{b}) &= \frac{1}{n} \sum_{k=1}^{n} b_k x_k \ , \quad (8) \\ \hat{\sigma}_0^2(\boldsymbol{b}) &= \frac{1}{n-1} \sum_{k=1}^{n} (b_k x_k - \hat{\mu}_0)^2 \ , \quad (9) \end{aligned}$$

If $B_k = b_k$ *then the conditional ML estimates are consistent (unbiased and their variance decreases for* $n \to \infty$*).*

The proof is rather standard and omitted here.

Theorem 3 *The* decision-directed *estimates are obtained by using hard-decision estimates* $\hat{\boldsymbol{b}}$ *instead of the true realization in the* conditional *ML estimator.*

$$\begin{aligned} \hat{b}_k &= \mathrm{sign}(x_k), \quad (k = 1, 2, \ldots, n), \quad (10) \\ \hat{\mu}_0 = \hat{\mu}_0(\hat{\boldsymbol{b}}) &= \frac{1}{n} \sum_{k=1}^{n} |x_k| \ , \quad (11) \\ \hat{\sigma}_0^2 = \hat{\sigma}_0^2(\hat{\boldsymbol{b}}) &= \frac{1}{n-1} \sum_{k=1}^{n} (|x_k| - \hat{\mu}_0)^2 \ , \quad (12) \end{aligned}$$

are asymptotically unbiased *in the limit* $\mu/\sigma \to \infty$.

The proof for $\hat{\mu}_0$ follows from the fact that $\mathrm{E}[\hat{\mu}_0] = \theta$ and taking the limit $\sigma \to 0$ in (3),

$$\lim_{\sigma \to 0} \theta = \mu \ . \quad (13)$$

The proof for $\hat{\sigma}_0^2$ follows from

$$\mathrm{E}\left[\hat{\sigma}_0^2\right] = \frac{1}{n-1} \sum_{k=1}^{n} \mathrm{E}\left[|x_k|^2\right] - \frac{n}{n-1} \mathrm{E}\left[\hat{\mu}_0^2\right] \quad (14)$$

which results in

$$\mathrm{E}\left[\hat{\sigma}_0^2\right] = \frac{n}{n-1}(\sigma^2 + \mu^2) - \frac{1}{n(n-1)} \sum_{k=1}^{n} \sum_{l=1}^{n} \mathrm{E}\left[|x_k x_l|\right] \ .$$

In the expection operation we need to distinguish between the cases $k = l$ and $k \neq l$ and we arrive at the following nice result

$$\mathrm{E}\left[\hat{\sigma}_0^2\right] = \sigma^2 + \mu^2 - \theta^2 = \varphi - \theta^2 \ . \quad (15)$$

After taking the limit $\mu \to \infty$,

$$\lim_{\mu \to \infty} (\varphi - \theta^2) = \sigma^2 + \lim_{\mu \to \infty} \epsilon(\mu, \sigma^2) \quad (16)$$

where

$$\epsilon(\mu, \sigma^2) = \mu^2 - \left(\mu \,\mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right) + \frac{2\sigma}{\sqrt{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)\right)^2 ,$$

we use the following bounds on the error function for $x > 0$

$$1 - \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} < \mathrm{erf}(x) < 1 - \frac{1}{x\sqrt{\pi}} \left(1 - \frac{2}{x^2}\right) \mathrm{e}^{-x^2} \quad (17)$$

for proving that $\lim_{\mu \to \infty} \epsilon(\mu, \sigma^2) = 0$.

Theorem 4 *The ML estimates for* (μ, σ^2) *satisfy the conditions*

$$\begin{aligned} \hat{\mu}_1 &= \frac{1}{n} \sum_{k=1}^{n} \frac{1 - \exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)}{1 + \exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)} |x_k| \\ &= \frac{1}{n} \sum_{k=1}^{n} x_k \tanh\left(\frac{\hat{\mu}_1 x_k}{\hat{\sigma}_1^2}\right) , \quad (18) \\ \hat{\sigma}_1^2 &= \frac{1}{n} \sum_{k=1}^{n} x_k^2 + \hat{\mu}_1^2 - \frac{2\hat{\mu}_1}{n} \sum_{k=1}^{n} x_k \tanh\left(\frac{\hat{\mu}_1 x_k}{\hat{\sigma}_1^2}\right) (19) \end{aligned}$$

This is easily proved by setting the gradient of (7) to zero. Note that the substitution of (18) in (19) reveals

$$\hat{\sigma}_1^2 = \frac{1}{n} \sum_{k=1}^{n} x_k^2 - \hat{\mu}_1^2 \quad (20)$$

which is intuitive.

Theorem 5 *The ML estimates* $(\hat{\mu}_1, \hat{\sigma}_1^2)$ *and the* decision-directed *estimates* $(\hat{\mu}_0, \hat{\sigma}_0^2)$ *are linked by conditions*

$$\begin{aligned} \hat{\mu}_1 &= \hat{\mu}_0 - \frac{2}{n} \sum_{k=1}^{n} \frac{\exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)}{1 + \exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)} |x_k| , (21) \\ \hat{\sigma}_1^2 &= \frac{n-1}{n} \left(\hat{\sigma}_0^2 + (\hat{\mu}_1 - \hat{\mu}_0)^2\right) + \\ &\quad \frac{4\hat{\mu}_1}{n} \sum_{k=1}^{n} \frac{\exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)}{1 + \exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)} |x_k| . \quad (22) \end{aligned}$$

input $x_1, \ldots, x_n$ {*observed sample*}
input N {*number of iterations*}
input $\hat{\mu}_0, \hat{\sigma}_0^2$ {*from* Eqs.(11) and (12) }
$m := \hat{\mu}_0$
$v := \hat{\sigma}_0^2$
for $i := 1$ **to** N

$$\begin{array}{lll} e_k & := & \exp(-2m|x_k|/v), \quad (\text{for } k = 1, \ldots, n) \\ w_k & := & \frac{e_k}{1+e_k}, \quad (\text{for } k = 1, \ldots, n) \\ \delta & := & \frac{1}{n} \sum_{k=1}^{n} w_k |x_k| \\ m & := & \hat{\mu}_0 - 2\delta, \\ v & := & \frac{n-1}{n}[\hat{\sigma}_0^2 + (m - \hat{\mu}_0)^2] + 4m\delta \end{array}$$

end
$\hat{\mu}_1 := m$
$\hat{\sigma}_1^2 := v$
output $\hat{\mu}_1, \hat{\sigma}_1^2$

Table 1. Batch Algorithm for BPSK

input $x_1, \ldots, x_n$ { *observed sample* }
input N { *number of iterations* }
input $\hat{\mu}_0, \hat{\sigma}_0^2$ {*from* Eqs.(11) and (12) }
$m := \hat{\mu}_0$
$v := \hat{\sigma}_0^2$
$E := \frac{1}{n} \sum_{k=1}^{n} x_k^2$
for $i := 1$ **to** N

$$\begin{array}{lll} e_k & := & \exp(-2m|x_k|/v), \quad (\text{for } k = 1, \ldots, n) \\ w_k & := & \frac{e_k}{1+e_k}, \quad (\text{for } k = 1, \ldots, n) \\ \delta & := & \frac{1}{n} \sum_{k=1}^{n} w_k |x_k| \\ m & := & \hat{\mu}_0 - 2\delta, \\ v & := & E - m^2 \end{array}$$

end
$\hat{\mu}_1 := m$
$\hat{\sigma}_1^2 := v$
output $\hat{\mu}_1, \hat{\sigma}_1^2$

Table 2. Simplified Batch Algorithm for BPSK

Corrollary 1 *The* decision-directed *estimates* $(\hat{\mu}_0, \hat{\sigma}_0^2)$ *define the following bounds on the ML estimates* $(\hat{\mu}_1, \hat{\sigma}_1^2)$*,*

$$0 \leq \hat{\mu}_1 \quad \leq \quad \hat{\mu}_0, \tag{23}$$

$$\frac{n-1}{n}\hat{\sigma}_0^2 \leq \hat{\sigma}_1^2 \quad \leq \quad \frac{n-1}{n}\left(\hat{\sigma}_0^2 + \hat{\mu}_0^2\right) + 2\hat{\mu}_0^2 \, . \tag{24}$$

These bounds are obtained from Eqs.(21) and (22) by using

$$0 \quad < \quad \frac{\exp(-x)}{1 + \exp(-x)} < \frac{1}{2}, \qquad \text{for } x > 0 \, . \tag{25}$$

4. EVALUATION OF THE ML ESTIMATES

In [4] a low bias algorithm for BPSK was proposed. The iteration procedure in [4] relies on a bisection method with the difficulty to find a suitable starting point for the iteration.

Proposition 3 *We interprete (21) and (22) as a 2×2 nonlinear system of fixed-point equations for the ML estimates. This results in the iterative algorithm in Table 1 for evaluating the ML estimates $\hat{\mu}_1, \hat{\sigma}_1^2$ given the decision-directed estimates $\hat{\mu}_0, \hat{\sigma}_0^2$.*

Proposition 4 *By means of eq. (20), the iteration scheme can be simplified to the Simplified Batch Algorithm given in Table 2.*

Theorem 6 *The Simplified Batch Algorithm in Table 2 converges unconditionally to the ML estimate.*

This is proved by showing that the 2×2 mapping defined by

$$\hat{\mu}_1(i+1) \quad = \quad f(\hat{\mu}_1(i), \hat{\sigma}_1(i)) \tag{26}$$

$$\hat{\sigma}_1^2(i+1) \quad = \quad g(\hat{\mu}_1(i), \hat{\sigma}_1^2(i)) \tag{27}$$

is *contracting*. Whether the map is contracting or not is determined by the inequality

$$D = \left| \det \begin{pmatrix} \frac{\partial}{\partial \hat{\mu}_1(k)} f & \frac{\partial}{\partial \hat{\sigma}_1^2(k)} f \\ \frac{\partial}{\partial \hat{\mu}_1(k)} g & \frac{\partial}{\partial \hat{\sigma}_1^2(k)} g \end{pmatrix} \right| < 1 \, . \tag{28}$$

After some manipulations, the convergence condition for the Simplified Batch Algorithm in Table 2 reduces to

$$D \quad = \quad \frac{\partial W}{\partial \hat{\sigma}_1^2} 2\hat{\mu}_1 = \frac{2\hat{\mu}_1}{n} \sum_{k=1}^{n} \frac{\hat{\mu}_1 x_k^2}{\hat{\sigma}_1^4 \cosh(\hat{\mu}_1 x_k / \sigma_1^2)^2} \tag{29}$$

$$\quad = \quad \frac{1}{n} \sum_{k=1}^{n} \frac{2a_k^2}{\cosh(a_k)^2} \tag{30}$$

with $\frac{\partial W}{\partial \hat{\sigma}_1^2} = \frac{1}{n} \sum_{k=1}^{n} \frac{2\hat{\mu}_1 |x_k| \exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2)}{\hat{\sigma}_1^4 (1+\exp(-2\hat{\mu}_1 |x_k| / \hat{\sigma}_1^2))^2} |x_k|$ and $a_k = \hat{\mu}_1 x_k / \sigma_1^2$. Now, we observe that $2a^2 / \cosh^2(a)$ is strictly less than 0.8785 for all real-valued a. Hence $D < 0.879$, and it is concluded that the iteration in Table 2 converges.

5. SIMULATION RESULTS

Suppose, our sample size is $n = 80$ and $\mu = 1$. We have carried out 200 Monte Carlo simulations (for each value of σ^2) of the approximate ML estimates $\hat{\mu}_1, \hat{\sigma}_1^2$ compared them to the decision-directed estimates $\hat{\mu}_0, \hat{\sigma}_0^2$. The approximate ML estimates are obtained from the iterative algorithm using $N = 5$ and $N = 20$ iterations, respectively. The results for the mean squared error are shown in Fig. 1 vs. the true signal to noise ratio γ. In this figure, the *decision-directed* estimates are denoted by (m0,v0) which are shown as blue circles ($\circ$), approximate ML estimates for $N = 5$ iterations are denoted by (m1,v1) which are shown as red crosses ($\times$), and approximate ML estimates for $N = 20$ iterations are denoted by (m2,v2) wich are shown as black diamonds ($\diamond$), respectively. The resulting estimates $\hat{\gamma}_\ell := \hat{\mu}_\ell^2/\hat{\sigma}_\ell^2$ for the SNR are compared in Fig. 2 for a range of *true* $\gamma = \mu^2/\sigma^2$ values.

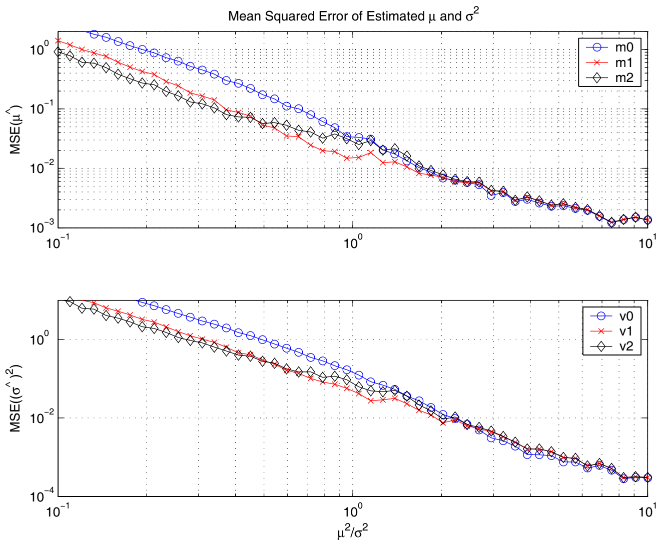

Fig. 1. Mean squared error of estimates for μ and σ^2 vs. γ

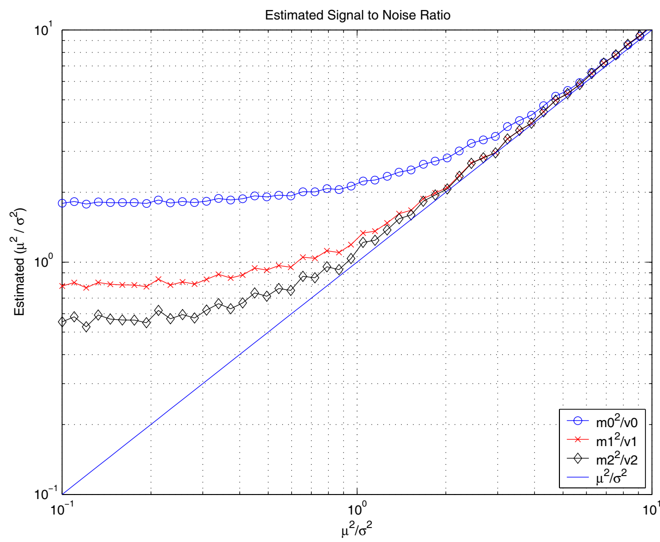

Fig. 2. Means of estimated signal to noise ratios. The decision-directed estimate $\hat{\mu}_0^2/\hat{\sigma}_0^2$ and two approximate ML-based estimates $\hat{\mu}_1^2/\hat{\sigma}_1^2$ obtained with the Batch Algorithm in Table 1 for $N = 5$ and $N = 20$ iterations are shown vs. γ.

6. CONCLUSION

We derive a fixed-point equation whose solution is the ML estimate of the amplitude of a BPSK modulated signal and the variance of the additive white Gaussian noise. An iterative algorithm is proposed for solving the fixed-point equation. Its convergence is proven and the resulting estimator performance is analyzed in simulations. Significant improvements in estimator fidelity are evident when compared to the *decision-directed* estimates even after small numbers of iterations.

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