CONSTRUCTION OF COHERENT SPACE TIME CODES FROM NON-COHERENT ONES

Oliver Henkel (henkel@hhi.fraunhofer.de)

Fraunhofer German-Sino Lab for Mobile Communications – MCI
Einsteinufer 37, 10587 Berlin, Germany

ABSTRACT

In this work, the natural geometric relation between the space time block code design if the channel is unknown at the receiver and its counterpart design if the receiver knows the channel is exploited, to establish an estimate on the corresponding diversities. This leads to a decomposition of code designs and splits the design problem into two complexity reduced sub tasks

1. INTRODUCTION

Since the multi antenna channel has been discovered as a source for high data rate communication a bulk of literature dealt with general capacity/performance analysis and code construction for the Rayleigh flat fading channel with no channel knowledge at the transmitter and known (resp. unknown) channel at the receiver, see e.g. [1], [2] (resp. [3], [4], [5]) for a small sample from the literature.

The dimensionality of the code symbols is determined by two system parameters, the block length n and the number k of transmit antennas. While the first one can be chosen rather large, upper bounded only by the coherence length of the channel, the second one is usually some small number due to hardware limitations. In order to keep the coding (and decoding) complexity low, a concentration on low dimensional unitary groups ($n = k$) as coding spaces took place soon ([6]), instead of taking the more general high dimensional Stiefel (known channel) and Grassmann manifolds (unknown channel) [3][4].

These general coding spaces consist of certain rectangular signal matrices of size $n \times k$ with arbitrary blocklength $n \geq k$. Inspired from [7], in [8] a general analysis of packings in the Stiefel and Grassmann manifold revealed, that there is an impact on the performance to expect if one chooses $n \gg k$, more precisely, the following proposition holds [8]):

Proposition 1.1.

Given the SNR $\rho \geq 1$, the rate $R = \frac{1}{n}\log|\mathcal{C}|$, $n \geq 2k$, and $D = k(2n-k)$ (known channel), resp. $D = 2k(n-k)$ (unknown channel). Then there exist space time block codes $\mathcal{C}$ with minimal distance $d^{\min}$ satisfying

$$d^{\min} \geq C\sqrt{\frac{n}{k}}\left(\frac{1}{2}\right)^{\frac{nR+1}{D}} \tag{1}$$

for some constant $C > 0$ depending on the channel knowledge at the receiver. Thus the performance increases monotonically at least proportionally to $\sqrt{\frac{n}{k}}$.

Having the common literature (see above) in mind, this result comes rather unexpected and further research effort seems promising. Moreover it becomes even more important when considering space frequency code design: Recent developments [9],[10] indicate, that the relevant coding spaces are certain subsets of (large dimensional) Stiefel and Grassmann manifolds. Thus considering these coding spaces in general may be of considerable importance for space frequency code designs.

In the present work it will be shown, how general space time block code designs can be decomposed into two 'smaller' pieces with reduced design complexity (Theorem 4.3), both already in the focus of current research. The achieved result can be seen as complementary to that of Kammoun and Belfiore [11], who presented a coding scheme for unknown channel space time block codes in terms of known channel ones, compare Remark 4.5 for further implications.

The key observation is the quite intuitive but technically not obvious diversity monotonicity (Proposition 4.2), which states that the performance of each unknown channel space time block code grows when considered as a known channel code. This can be interpreted as to be due to some higher resolution of the known channel receiver compared to its unknown channel counterpart, reflecting the information theoretic relation between the system designs.

Before we come to these results in section 4, section 2 first introduces the channel model and basic properties of the coding spaces before the diversity as our fundamental performance measure will be defined in section 3. Note that thanks to the geometric picture emphasized here the actual proofs of the propositions in section 4 seem quite elementary, and they will be sketched here only. Indeed the statements follow once the principal fibre structure (7) relating the coding spaces is exploited. In this sense it is a favour of this work to demonstrate the power and spirit of geometrical methods in space time coding.

2. CHANNEL MODEL AND CODING SPACES

We consider the Rayleigh flat fading MIMO (multiple input multiple output) channel without channel knowledge at the transmitter and maximum likelihood decoding at the receiver as described in [3] (with normed expected power $\sum_j \mathsf{E}|s_{ij}|^2 = 1$ per time step, $i = 1, \ldots, n$, E denotes expectation):

$$X = \sqrt{\rho}\, SH + W \tag{2}$$

$S = (s_{ij}) \in \mathbb{C}^{n\times k}$, $H \in \mathbb{C}^{k\times k'}$, $X, W \in \mathbb{C}^{n\times k'}$, whereas n denotes the coherence time of the channel (resp. the block length of the signals), k, k' denote the number of transmit, resp. receiver antennas, $W \sim$ i.i.d. $\mathcal{CN}(0,1)$ is the noise, $H \sim$ i.i.d. $\mathcal{CN}(0,1)$ the channel matrix and S, X denote the transmitted, resp. received signal with SNR (signal to noise ratio) ρ.

Nowadays it is common sense (see Hochwald and Marzetta [3, Theorem 1 and 2]) that in the high SNR regime and under the assumption of an equal power profile signals $S \in \mathbb{C}^{n\times k}$ of the form

$$S = \sqrt{\frac{n}{k}}\,\Phi\,, \quad \Phi^\dagger\Phi = \mathbf{1}_k \tag{3}$$

represent optimal transmission strategies for a variety of scenarios. Due to the unitarity condition ('unitary space time modulation', see [4]) $\Phi^\dagger\Phi = \mathbf{1}_k$ ($\mathbf{1}_k$ denotes the $k\times k$ identity matrix) imposed on the column vectors of $\Phi \in \mathbb{C}^{n\times k}$, the coding space is essentially the complex Stiefel manifold $V^{\mathbb{C}}_{k,n}$ of unitary k-frames in $\mathbb{C}^n$. Moreover, if the channel is unknown at the receiver the mutual information (ergodic in the channel realisations) depends only on the subspace in $\mathbb{C}^n$ spanned by the columns of Φ, not on the k-frame itself [12]. Thus in this case we obtain Φ as being an element of the complex Grassmann manifold of k-dimensional linear subspaces of $\mathbb{C}^n$. Let us explore the coding spaces in more detail:

Known channel — The Stiefel manifold $V^{\mathbb{C}}_{k,n}$: The Stiefel manifold $V^{\mathbb{C}}_{k,n} := \{\Phi \in \mathbb{C}^{n\times k} \,|\, \Phi^\dagger\Phi = \mathbf{1}\}$ is canonically a homogeneous space $\big(U(n) \xrightarrow{\pi} V^{\mathbb{C}}_{k,n};\, U(n-k)\big)$: The canonical left multiplication of k-frames in $\mathbb{C}^n$ by unitary $n\times n$ matrices transforms each pair of k-frames into each other. Thus the action of the unitary group $U(n)$ on $V^{\mathbb{C}}_{k,n}$ is transitive and establishes the canonical diffeomorphism $V^{\mathbb{C}}_{k,n} \cong U(n)\big/\left(\begin{smallmatrix}1 & \mathbf{0}\\ \mathbf{0} & U(n-k)\end{smallmatrix}\right)$.

A code $\mathcal{C}^V$ for the known channel model is given by a discrete set $\mathcal{C}^V = \{\sqrt{\frac{n}{k}}\,\Phi\} \subset \sqrt{\frac{n}{k}}\,V^{\mathbb{C}}_{k,n}$. At the receiver the maximum likelihood decision reads (see [4])

$$\Phi_{\mathrm{ML}}: \left\|X - \sqrt{\rho\tfrac{n}{k}}\,\Phi_{\mathrm{ML}}H\right\|_{\mathrm{F}} \le \left\|X - \sqrt{\rho\tfrac{n}{k}}\,\Phi H\right\|_{\mathrm{F}} \;\forall_{\Phi\in\mathcal{C}} \tag{4}$$

whereas $X = \sqrt{\rho\frac{n}{k}}\,\Psi H + W$ is the received signal.

Unknown channel — The Grassmann manifold $G^{\mathbb{C}}_{k,n}$: The Grassmann manifold $G^{\mathbb{C}}_{k,n}$ of all k-dimensional linear subspaces $\langle\Phi\rangle$ of $\mathbb{C}^n$, $\Phi \in V^{\mathbb{C}}_{k,n}$, also carries the structure of a $U(n)$-normal homogeneous space $\big(U(n) \xrightarrow{\pi} G^{\mathbb{C}}_{k,n};\, U(k)\times U(n-k)\big)$ by neglecting not only the orthogonal complement of the column vectors in Φ (which has been done for $V^{\mathbb{C}}_{k,n}$) but also the particular choice of the spanning k-frame. This leads to the representation $G^{\mathbb{C}}_{k,n} \cong U(n)\Big/\left(\begin{smallmatrix}U(k) & \mathbf{0}\\ \mathbf{0} & U(n-k)\end{smallmatrix}\right)$. To simplify matters let us assume $k \le n/2$ (otherwise switch to orthogonal complements). Then each two elements $\langle\Phi\rangle, \langle\Psi\rangle \in G^{\mathbb{C}}_{k,n}$ are separated by the k stationary (or principal) angles $0 \le \vartheta_1 \le \cdots \le \vartheta_k \le \pi/2$ (defined successively by the critical values of $(v,w) \longmapsto \arccos|<v,w>|$ for unit vectors in $\langle\Phi\rangle$, resp. $\langle\Psi\rangle$), with (any representing k-frame will do)

$$\cos\vartheta_i = \sigma_i(\Phi^\dagger\Psi) \tag{5}$$

An important application of stationary angles on some given pair $\langle\Phi\rangle, \langle\Psi\rangle$ with principal angles (ϑ_i) is, that due to the transitivity of the unitary group action there exist an unitary U, such that Ψ (say) can always be translated into $\Psi_0 := \left(\begin{smallmatrix}\mathbf{1}\\ \mathbf{0}\end{smallmatrix}\right) = U\Psi$ and in $U\langle\Phi\rangle = \langle U\Phi\rangle$ one can choose a basis such that we end up with the canonical representing k-frames

$$\Psi_0 = (\mathbf{1}, \mathbf{0})^T\,, \quad \Phi_0 = ((\cos\vartheta_i), (\sin\vartheta_i), \mathbf{0})^T \tag{6}$$

(where $(\cos\vartheta_i) := \mathrm{diag}(\cos\vartheta_i)_{i=1,\dots,k} \in \mathbb{R}^{k\times k}$) for the translated spaces $\langle\Psi_0\rangle = U\langle\Psi\rangle$, $\langle\Phi_0\rangle = U\langle\Phi\rangle$.

The principal fibre structure P^V_G: The natural relationship between the homogeneous spaces $V^{\mathbb{C}}_{k,n}$ and $G^{\mathbb{C}}_{k,n}$ is subsumed in the canonical principal fibre bundle structure

$$P^V_G := \left(V^{\mathbb{C}}_{k,n} \xrightarrow{\pi^V_G} G^{\mathbb{C}}_{k,n};\, U(k)\right) \tag{7}$$

which (locally) embeds $G^{\mathbb{C}}_{k,n}$ into $V^{\mathbb{C}}_{k,n}$ by choosing a representing k-frame Φu which spans the subspace $\langle\Phi\rangle$. However there remains the freedom of multiplication with arbitrary unitary matrices $u \in U(k)$ from the right, and for practical applications it is necessary to specify a unique choice for Φ and u, given $\langle\Phi\rangle$ (simultaneously for all $\langle\Phi\rangle \in G^{\mathbb{C}}_{k,n}$, not only for pairs as in (6)). But locally this can always be achieved and we do not want to go into details here.

So finally we can consider codes $\mathcal{C}^G \subset \sqrt{\frac{n}{k}}G^{\mathbb{C}}_{k,n}$ always as discrete subsets of $\sqrt{\frac{n}{k}}V^{\mathbb{C}}_{k,n}$ and the maximum likelihood criterion for the unknown channel receiver reads now ([4])

$$\Phi_{\mathrm{ML}}: \left\|\sqrt{\rho\tfrac{n}{k}}\,\Phi^\dagger_{\mathrm{ML}}X\right\|_{\mathrm{F}} \ge \left\|\sqrt{\rho\tfrac{n}{k}}\,\Phi^\dagger X\right\|_{\mathrm{F}} \quad \forall_{\Phi\in\mathcal{C}} \tag{8}$$

whereas $X = \sqrt{\rho\frac{n}{k}}\,\Psi H + W$ is the received signal.

3. PERFORMANCE ANALYSIS: DIVERSITY

The pairwise error probability P of mistaking the symbol Φ for Ψ (kept fixed in this section) at the receiver can be (Chernov) upper bounded as [4]

$$P \le \frac{1}{2}\left(\prod_{i=1}^{k}\left[1 + \varrho\,\sigma_i^2\right]\right)^{-k'} \tag{9}$$

whereas (known channel)

$$\varrho = \bar\varrho := \frac{\rho n}{4k} \tag{10}$$

$$\sigma_i = \bar\sigma_i := \sigma_i(\Phi - \Psi) \in [0,2] \tag{11}$$

satisfying the invariance property

$$\sigma_i(U(\Phi-\Psi)v) = \sigma_i(\Phi-\Psi) \quad \forall_{U\in U(n),\, v\in U(k)} \tag{12}$$

respectively (unknown channel)

$$\varrho = \underline{\varrho} := \frac{\bar\varrho^2}{\bar\varrho + 1/4} = \frac{(\rho\frac{n}{k})^2}{4(1+\rho\frac{n}{k})} \tag{13}$$

$$\sigma_i = \sqrt{1 - \underline{\sigma}_i^2}\,, \quad \underline{\sigma}_i := \sigma_i(\Phi^\dagger\Psi) = \cos\vartheta_i \in [0,1] \tag{14}$$

satisfying (in correspondence to (6))

$$\sigma_i((U\Phi v)^\dagger(U\Psi w)) = \sigma_i(\Phi^\dagger\Psi) \quad \forall_{U\in U(n),\, v,w\in U(k)} \tag{15}$$

The term in parentheses in (9) is called (pairwise) diversity $\mathcal{D}iv$, and we take it as our basic performance measure for codes. With the elementary symmetric polynomials defined by $\mathrm{sym}^k_0 := 1, \mathrm{sym}^k_j(x_1,\dots,x_k) = \sum_{I_j\in S^k_j} x_{I_j} = \sum_{I_j\in S^k_j} x_{i_1}\cdots x_{i_j}$ (with

$S_j^k := \{(i_1,\dots,i_j) \in \mathbb{N}^j \,|\, 1 \le i_1 < \cdots < i_j \le k\})$, $j = 1,\dots,k$, and with the notation

$$s_j := \mathrm{sym}_j^k(\sigma_1^2,\dots,\sigma_k^2) = \sum_{I_j\in S_j^k} s_{I_j}^2 \tag{16}$$

we find generically

$$\mathcal{D}iv := \sum_{i=0}^{k} s_i \varrho^i \tag{17}$$

The first and highest order term of $\mathcal{D}iv$ are the well known diversity sum d (d for 'distance') and diversity product p. While the maximum likelihood receivers (4), (8) measure d directly, the diversity product becomes important as the leading term in the diversity.

Note that the known channel diversity $\overline{\mathcal{D}iv}$ is formally similar to the unknown channel diversity $\underline{\mathcal{D}iv}$ by (17), resp. (7), but the constituting singular values (11), (14) reflect the underlying topological structures induced by the maximum likelihood receivers (4), (8). And these structures are entirely distinct.

4. EMBEDDING PROPERTIES

Now let us investigate the relation between the unknown and known channel diversity quantities. From the information theoretic inequality $I(X;S) \le I((X,H);S)$ between the corresponding mutual informations we expect such a relation satisfied by the diversity. The ranges for $\bar\sigma$ (11) and $\underline{\sigma}$ (14) indicate, that the known channel receiver may benefit from some higher 'resolution', but if and how this carries over to the diversity is not obvious and requires a rigorous proof. The investigations of this section give an affirmative answer to that conjecture.

By a slight abuse of notation let us define the fibre minima of $\bar s_{I_j}$ with respect to the fibres of P_G^V (7) as

$$\bar{s}\mathrm{dist}_{I_j}(\Phi,\Psi) := \min_{\substack{\Phi\in\pi_G^{V\,-1}(\langle\Phi\rangle)\\ \Psi\in\pi_G^{V\,-1}(\langle\Psi\rangle)}} \bar s_{I_j}(\Phi,\Psi) \tag{18}$$

$\bar{s}\mathrm{dist}_j(\Phi,\Psi) := \sum_{I_j\in S_j^k} \bar{s}\mathrm{dist}_{I_j}^2(\Phi,\Psi)$, and $s^{\min} := \min_{\mathcal{C}\times\mathcal{C}} s$ for functions $s : \mathcal{C} \longrightarrow \mathbb{R}$. Then the following lemma holds:

Lemma 4.1.

Let $\langle\Phi\rangle, \langle\Psi\rangle \in G_{k,n}^{\mathbb{C}}$ separated by principal angles $\vartheta_1,\dots,\vartheta_k$. Then successively

$$\forall_{I_j\in S_j^k}\ \bar{s}\mathrm{dist}_{I_j}(\Phi,\Psi) = \sqrt{2^j \prod_{i\in I_j}(1-\cos\vartheta_{k-i+1})} \tag{19}$$

$$\forall_j\ \underline{s}_j(\Phi,\Psi) \le \bar{s}\mathrm{dist}_j(\Phi,\Psi) \le \bar s_j(\Phi,\Psi) \tag{20}$$

$$\underline{s}_j^{\min} \le \bar{s}\mathrm{dist}_j^{\min} \le \bar s_j^{\min} \tag{21}$$

Outline of proof. [1]
(19) follows by matrix analysis applied to $\bar\sigma$, $\underline{\sigma}$ exploiting the degrees of freedom (12), (15) and (6). The inequalities (20) are a simple consequence of $2\cos\alpha - \cos^2\alpha \le 1$ in $[0, \frac{\pi}{2}]$, and (21) is straight forward. ∎

[1] A more detailed proof can be found in [13]

From Lemma 4.1 together with $\frac{1}{2}\bar\varrho \le \underline{\varrho} \le \bar\varrho$, if $\rho \ge 1$ ($n \ge k$ is understood), it follows

Proposition 4.2.

For any pair $\langle\Phi\rangle, \langle\Psi\rangle \in G_{k,n}^{\mathbb{C}}$ and each fixed $\rho \ge 1$ holds

$$\underline{\mathcal{D}iv}(\Phi,\Psi) \le \overline{\mathcal{D}iv}(\Phi,\Psi) \tag{22}$$

We conclude, that the known channel maximum likelihood receiver applied to $\mathcal{C}^G$ has at least the diversity as the unknown channel receiver, the diversity grows.

Having explored the relationship of the embedding $G_{k,n}^{\mathbb{C}} \subset V_{k,n}^{\mathbb{C}}$ let us come to a somewhat complementary scenario, which offers the possibility of coding complexity reduction: Consider a single fibre over $\langle\Phi\rangle$. Then, by $\Phi w = (\Phi, \Phi^\perp)\binom{w}{0}$, there holds a special kind of 'vertical' left invariance, namely

$$\bar s_{I_j}(\Phi u, \Phi v) = \bar s_{I_j}^U(u,v)\,, \qquad \forall_{I_j\in S_j^k}\,,\ \forall_{u,v\in U(k)} \tag{23}$$

where the right hand side is evaluated in $U(k) = V_{k,k}^{\mathbb{C}}$. Analogously we define for the special case $n = k$: $\bar\varrho^U := \frac{\varrho}{4}$, $\bar s_j^U := \sum_{I_j\in S_j^k}(\bar s_{I_j}^U)^2$, and $\overline{\mathcal{D}iv}^U := \sum_i \bar s_i^U(\bar\varrho^U)^i$. Then we arrive at

Theorem 4.3.

Given codes $\mathcal{C}^G \subset \sqrt{\frac{n}{k}}\,G_{k,n}^{\mathbb{C}} \subset \sqrt{\frac{n}{k}}\,V_{k,n}^{\mathbb{C}}$ and $\mathcal{C}^U \subset U(k)$, then the composed code $\mathcal{C}^V \subset \sqrt{\frac{n}{k}}\,V_{k,n}^{\mathbb{C}}$ given by

$$\mathcal{C}^V := \mathcal{C}^G \cdot \mathcal{C}^U = \left\{\Phi u \,|\, \Phi\in\mathcal{C}^G, u\in\mathcal{C}^U\right\} \tag{24}$$

satisfies

$$\bar s_j^{\min} \ge \min\{\bar{s}\mathrm{dist}_j^{\min}, \bar s_j^{U\,\min}\}\,, \qquad \forall_{j=1,\dots,k} \tag{25}$$

and for $\rho \ge 1$

$$\overline{\mathcal{D}iv}^{\min} \ge \min\{\underline{\mathcal{D}iv}^{\min}, \widetilde{\overline{\mathcal{D}iv}}^{U\,\min}\} \tag{26}$$

holds, whereas $\widetilde{\overline{\mathcal{D}iv}}^U := \sum_i (\frac{n}{k})^i \bar s_i^U(\bar\varrho^U)^i$ (thus the power constraint factor $\sqrt{\frac{n}{k}}$ sharpens the estimate).

Therefore the code design splits up into two parts: Codes $\mathcal{C}^G$ represent the familiar coding problem for the unknown channel corresponding to $\sqrt{\frac{n}{k}}\,G_{k,n}^{\mathbb{C}}$, which has smaller dimension as the general problem in $\sqrt{\frac{n}{k}}\,V_{k,n}^{\mathbb{C}}$. The code $\mathcal{C}^U$ represents a coding problem for the known channel in $U(k) = V_{k,k}^{\mathbb{C}}$, contributing the dimensions left by $V_{k,n}^{\mathbb{C}} \cong G_{k,n}^{\mathbb{C}} \times U(k)$ locally. So both parts represent a somewhat smaller coding problem, reducing therefore the complexity of the overall problem.

Remark 4.4.

A related question arises, when one considers the task of given a code $\mathcal{C}^U$, does there exist a code $\mathcal{C}^V$ with the same rate but better performance than $\mathcal{C}^U$? Concerning the diversity sum d a partial answer gives [8]: The transmit power constraint sets the requirement $\sqrt{\frac{n}{k}}\bar d^{\min} \ge \bar d^{U\,\min}$. Since there exist a monotonically increasing lower bound for $\bar d^{\min}$ when $\frac{n}{k}$ grows

(Proposition 1.1) this requirement can be certainly fulfilled. This again emphasises the need for coding strategies in the general coding spaces $V^{\mathbb{C}}_{k,n}$, $G^{\mathbb{C}}_{k,n}$, n larger than k. However, it remains an open question, whether one can achieve the goal by composed codes of the form $\mathcal{C}^V = \mathcal{C}^G \cdot \mathcal{C}^U$.

Remark 4.5.

A conceptual simple (but computational complex) embedding of $G^{\mathbb{C}}_{k,n}$ into $V^{\mathbb{C}}_{k,n}$ is given by the parametrisation of $G^{\mathbb{C}}_{k,n}$ with horizontal tangents $X^H = \begin{pmatrix} \mathbf{0} & -B^\dagger \\ B & \mathbf{0} \end{pmatrix}$, $B \in \mathbb{C}^{(n-k)\times k}$ in its total space $U(n)$. In a recent article [11] it has been shown, that coding for the unknown channel is under certain assumptions equivalent to coding on the horizontal tangent space, with respect to the known channel diversity in $V^{\mathbb{C}}_{k,n-k}$. Combining that with Theorem 4.3 we can roughly state this correspondence as $V^{\mathbb{C}}_{k,n-k} \subset G^{\mathbb{C}}_{k,n} \subset V^{\mathbb{C}}_{k,n}$, which gives rise to a sequence $\ldots \to \mathcal{C}^V_i \to \mathcal{C}^G_{i+1} \to \mathcal{C}^V_{i+1} \to \ldots$ of codes with increasing block length $i \cdot k$, $i = 1, 2, \ldots$

5. CONCLUSION

It has been shown, that the diversity, taken as a performance measure, grows when an unknown channel space time block code is used in the known channel scenario (Proposition 4.2). This result turned out to be due to the various invariance properties satisfied by the diversity, though tied to distinct underlying topologies of the coding spaces induced by the maximum likelihood receiver metrics.

Exploiting some further 'vertical' invariance, embeddings of both $G^{\mathbb{C}}_{k,n}$ and $U(k)$ into $V^{\mathbb{C}}_{k,n}$ led to the decomposition of codes on $V^{\mathbb{C}}_{k,n}$ into 'smaller' pieces in $G^{\mathbb{C}}_{k,n}$ and $U(k)$ (Theorem 4.3), both of them being already in the focus of current research. The other way round, given an unknown channel space time code and a 'small' known channel code, the performance of the resulting (larger dimensional) product code on $V^{\mathbb{C}}_{k,n}$ is lower bounded by the diversity expressions stated in the theorem. Thus the design complexity has been reduced to the smaller problems on $G^{\mathbb{C}}_{k,n}$ and $U(k)$. Together with Proposition 1.1 this opens the door to potentially high performing space time block codes, when $n \gg k$.

Following this line of thought, this work could be seen as a second step towards a geometry based analysis of general space time block codes inspired by [8]. It demonstrates the power of geometrical methods in space time coding theory. As already indicated in the introduction, the results may be of some importance in the context of space frequency codes also. What remains is the challenge of effective high dimensional code construction (especially for the unknown channel) with low complexity decoding properties, which will be addressed in a future work.

6. REFERENCES

[1] V. Tarokh, N. Seshadri, and A.R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, 1998.

[2] V. Tarokh, H. Jafarkhani, and A.R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, 1999.

[3] B.M. Hochwald and T. Marzetta, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 139–157, 1999.

[4] B.M. Hochwald and T. Marzetta, "Unitary space-time modulation for multiple-antenna communications in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 46, pp. 543–565, 2000.

[5] V. Tarokh, "Existence and construction of non-coherent unitary space-time codes," preprint, url: http://www.mit.edu/~vahid/sample.html.

[6] S.M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE Journal on Select. Areas in Communications*, vol. 16, pp. 1451–1458, 1998.

[7] A. Barg and D.Y. Nogin, "Bounds on packings of spheres in the Grassmann manifold," *IEEE Trans. Inform. Theory*, vol. 48, pp. 2450–2454, 2002.

[8] O. Henkel, "Sphere packing bounds in the Grassmann and Stiefel manifolds," *IEEE Trans. Inform. Theory*, (to be published).

[9] H. Bölcskei, M. Borgmann, and A.J. Paulraj, "Space-frequency coded MIMO-OFDM with variable multiplexing-diversity tradeoff," *submitted to IEEE Trans. Inform. Theory*, 2004.

[10] H. Bölcskei and M. Borgmann, "Code design for non-coherent MIMO-OFDM systems," *submitted to IEEE Trans. Inform. Theory*, 2004.

[11] I. Kammoun and J.-C. Belfiore, "A new family of Grassmann space-time codes for non-coherent MIMO systems," *IEEE Comm. Letters*, vol. 7, no. 11, pp. 528–530, 2003.

[12] L. Zheng and D.N.C. Tse, "Communication on the Grassmann manifold: a geometric approach to the noncoherent multiple-antenna channel," *IEEE Trans. Inform. Theory*, vol. 48, pp. 359–383, 2002.

[13] O. Henkel, "Geometrical relations between space time block code designs and complexity reduction," *submitted to IEEE Trans. Inform. Theory*, 2003.