OPTIMAL NORM FORM INTEGER SPACE-TIME CODES FOR TWO ANTENNA MIMO SYSTEMS

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ABSTRACT

In this paper, we introduce the definitions of a full diversity integer generating matrix and the corresponding norm form space-time code for MIMO systems. Subject to a power constraint, we characterize all full diversity integer generating matrices with the first three largest gains in the Gaussian integer ring and the Eisenstein integer ring for two transmitter antennas. Using this generating matrix family to separately design space-time codes layer by layer for two transmitter antenna and two receiver antenna MIMO systems, we obtain the optimal norm form integer space-time codes both in the Gaussian integer ring and the Eisentein integer ring in the sense of maximizing the minimum determinant of codeword matrices. As a consequence, we prove that the golden code constructed by Dayal and Varanasi, Belfiore, Rekaya and Viterbo is optimal in the Gaussian integer ring. Also, we find the optimal code in the Eisentein integer ring, the coding gain of which is greater than that of the golden code.

1. INTRODUCTION

In this paper, we focus on a coherent MIMO system with two transmitter antennas and two receiver antennas. For such a system, Damen, Tewfik and Belfiore [1] first constructed a full rate full diversity linear space-time block code without information loss for M-ary QAM signals. Recently, this result was generalized to any number of transmitter antennas [2], [3], [4] and to a rectangular linear dispersion code case [5], [6]. The main issue on these current designs is that the coding gain vanishes rapidly as the constellation size increases. Therefore, full rate full diversity non-vanishing space-time code designs have recently drawn much attention [7], [8], [9], where the coding gain for two transmitter antenna and two receiver antenna MIMO systems is constant in the whole Gaussian integer ring $\mathbb{Z}[j]$. Lately, this non-vanishing codes have been extended to a MIMO system with three, four and six transmitter antennas [10], [11], [12], [13].

In this paper, we introduce the definitions of a full diversity integer generating matrix and the corresponding norm form integer space-time code for MIMO systems. Subject to a power constraint, we characterize all full diversity integer generating matrices with the first three largest gains in the Gaussian integer ring and the Eisenstein integer ring for two transmitter antennas. Utilizing this generating matrix family to design norm form integer space-time codes for two transmitter antenna and two receiver antenna MIMO systems, we obtain the optimal codes in the sense of maximizing the minimum determinant of codeword matrices. As a consequence, we prove that the codes constructed by Dayal and Varanasi, Belfiore, Rekaya and Viterbo are optimal among all integer generating matrix family with the same power in the Gaussian integer ring[1]. Also, we find the optimal codes in the Eisentein integer ring $\mathbb{Z}[\zeta_3]$, the error performance of which is better than the golden code.

Notation: Throughout this paper, we use the following notation: Matrices are denoted by uppercase boldface characters (e.g., $\mathbf{A}$), while column vectors are denoted by lowercase boldface characters (e.g., $\mathbf{b}$). The i-th entry of $\mathbf{b}$ is denoted by b_i. The columns of an $M \times N$ matrix $\mathbf{A}$ are denoted by $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N$. The Hermitian transpose of $\mathbf{A}$ (i.e., the conjugate transpose of $\mathbf{A}$) is denoted by $\mathbf{A}^H$. $\mathbb{Z}$ denotes the rational integer ring; $\mathbb{C}$ denotes the field of complex number; $j = \sqrt{-1}$; $\zeta_m = \exp\left(\frac{j2\pi}{m}\right)$; $\mathbb{Z}[\zeta_m]$ denotes the cyclotomic ring generated by $\mathbb{Z}$ and the cyclotomic number ζ_m. For simplicity, we also use notation $\mathbb{I}_n$ to denote the Gaussain integer ring $\mathbb{Z}[j]$ or the Eisentein integer ring $\mathbb{Z}[\zeta_3]$.

2. CHARACTERIZATION OF INTEGER GENERATING MATRICES WITH FULL DIVERSITY

In this section, we first define a full diversity integer generating matrix and then we characterize all such matrices with the first three largest coding gains in the Gaussian integer ring and the Eisentein ring subject to a power constraint.

2.1. Definition of full diversity integer generating matrices

First, we introduce the definition of a full diversity integer generating matrix.

Definition 1 *Let* $\mathbf{s} \in \mathbb{I}_n^M$. *An* $M \times M$ *matrix* $\mathbf{G}$ *is said to be an integer generating matrix if there exists a nonzero constant* $G \in \mathbb{C}$ *such that for any* $\mathbf{s} \in \mathbb{I}_n^M$, $\prod_{k=1}^M x_k = G\alpha$, *where* $\mathbf{x} = \mathbf{G}\,\mathbf{s}$ *and* $\alpha \in \mathbb{I}_n$. *This generating matrix* $\mathbf{G}$ *is said to be of full diversity if for any* $\mathbf{s} \in \mathbb{I}_n^M \setminus 0$, $\prod_{k=1}^M x_k \neq 0$. *The absolute value* $|G|$ *of constant* G *is called the gain of the integer generating matrix* $\mathbf{G}$. ∎

A typical integer generating matrix example is the cyclotomic generating matrix developed in [14], [15], [16], [17], [18] for a specific

[1] Dayal and Varanasi [8] proved that their code; i.e., the golden code, is optimal among all real rotation generating matrices.

number M. In particular, when $M=2$, a full diversity integer cyclotomic unitary generating matrix is given by

$$\mathbf{G}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}1 & \exp(\frac{\pi j}{4}) \\ 1 & -\exp(\frac{\pi j}{4})\end{array}\right) \tag{1}$$

Here, a natural question is whether such a generating matrix for an arbitrarily given number of M exists and whether all such generating matrices can be characterized if exists? In the next subsection, we will characterize all such two by two matrices with the first, second and third largest gain in the Gaussian integer ring and the Eisentein ring subject to a power constraint.

2.2. Parameterization of full diversity integer generating matrices

In this subsection, we characterize all two by two full diversity integer generating matrices. Let $\mathbf{G}=\left(\begin{array}{cc}a & b \\ c & d\end{array}\right)$. Now we enforce a power constraint and the full diversity constraint on the generating matrix to design the two by two generating matrix $\mathbf{G}$ such that the gain $|G|$ is maximized. Our problem can be formally stated as

Problem 1 *Find a two by two integer generating matrix* $\mathbf{G}$ *such that the gain* $|G|$ *is maximized subject to the following two constraints:*

1. *Power constraint:*

$$\operatorname{tr}\left(\mathbf{G}\mathbf{G}^{\mathrm{H}}\right)=|\mathrm{a}|^2+|\mathrm{b}|^2+|\mathrm{c}|^2+|\mathrm{d}|^2=\mathrm{p} \tag{2}$$

2. *Full diversity constraint of the integer generating matrix* $\mathbf{G}$. ■

We can prove that Problem 1 is equivalent to the following optimization problem.

Formulation 1 *Find integers m_1, m_2 and m_3 in the ring $\mathbb{I}_n$ such that*

$$\max\ |G|^2=\frac{p^2}{2\left(2(|m_1|^2+|m_2|^2)+|m_3|^2+|\Delta|\right)} \tag{3}$$

subject to the full diversity constraint that $\Delta=m_3^2-4m_1m_2$ is square-free in the integer ring $\mathbb{I}_n$. ■

We are now in a position to formally state our main results.

Theorem 1 *For the Gaussian integer ring $\mathbf{Z}[j]$, the optimal solution of Problem 1 with the first three largest gains, denoted by $|G_1|$, $|G_2|$ and $|G_3|$, respectively, can be characterized as follows:*

(1) The first largest gain is $|G_1|=\frac{p}{4}$ and the corresponding generating matrices are

$$\mathcal{G}_{11}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & e^{j\varphi} \\ 1 & -e^{j\varphi}\end{array}\right), \tag{4}$$

$$\mathcal{G}_{12}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & e^{-j(\theta+\phi)} \\ 1 & e^{j(\theta+2\phi)}\end{array}\right), \tag{5}$$

$$\mathcal{G}_{13}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & -je^{-j(\theta+\phi)} \\ 1 & je^{j(\theta+2\phi)}\end{array}\right), \tag{6}$$

where $\varphi=\pm\frac{\pi}{4},\pm\frac{3\pi}{4}$, $\theta=\pm\frac{\pi}{3},\pm\frac{2\pi}{3}$, $\phi=0,\pi,\pm\frac{\pi}{2}$ and α and β are arbitrary real numbers.

(2) The second largest gain is $|G_2|=\frac{p}{\sqrt{10+2\sqrt{17}}}$ and matrices are

$$\mathcal{G}_2=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}\sqrt{\frac{2}{1+|t|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|t|^2}{5+\sqrt{17}}}e^{j\beta}\end{array}\right)\left(\begin{array}{cc}t & 1 \\ m_1t^{-1} & m_2\end{array}\right)$$

where α and β are arbitrary real numbers, m_1 and m_2 are

$$\left\{\begin{array}{l}m_1=1 \\ m_2=-j\end{array}\right.,\ \left\{\begin{array}{l}m_1=-1 \\ m_2=j\end{array}\right.,\ \left\{\begin{array}{l}m_1=j \\ m_2=-1\end{array}\right.,\ \left\{\begin{array}{l}m_1=-j \\ m_2=1\end{array}\right.$$

and t is determined by $t=\frac{m_3+\sqrt{m_3^2-4m_1m_2}}{2m_2}$ with $m_3=\pm1,\pm j$.

(3) The third largest gain is $|G_3|=\frac{p}{2\sqrt{5}}$ and the generating matrices are given by

$$\mathcal{G}_{31}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}\sqrt{\frac{2}{1+|\tau|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\tau|^2}{10}}e^{j\beta}\end{array}\right)\left(\begin{array}{cc}\tau & 1 \\ -\tau^{-1} & m^2\end{array}\right)$$

$$\mathcal{G}_{32}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}\sqrt{\frac{2}{1+|\tau|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\tau|^2}{10}}e^{j\beta}\end{array}\right)\left(\begin{array}{cc}\tau & -j \\ \tau^{-1} & m^2j\end{array}\right),$$

where α and β are arbitrary real numbers and τ is determined by $\tau=\frac{\pm1\pm\sqrt{5}}{2m}$ with $m=\pm1,\pm j$. ■

Theorem 2 *For the Eisentein integer ring, the optimal solution of Problem 1 with the first three largest gains, denoted by $|E_1|$, $|E_2|$ and $|E_3|$, respectively, can be parameterized as follows:*

(1) The first largest gain is $|E_1|=\frac{p}{4}$ and the generating matrices are

$$\mathcal{E}_{11}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & \mp mj \\ 1 & \pm mj\end{array}\right),$$

$$\mathcal{E}_{12}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & \mp m\zeta_3^2j \\ \zeta_3 & \pm m\zeta_6j\end{array}\right),$$

$$\mathcal{E}_{13}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & \mp m\zeta_{12}^{-1}j \\ \zeta_6 & \pm m\zeta_{12}j\end{array}\right),$$

$$\mathcal{E}_{14}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & \mp m\zeta_{12}^{-1}j \\ -\zeta_6 & \pm m\zeta_{12}j\end{array}\right),$$

$$\mathcal{E}_{15}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & m\zeta_3e^{j\theta} \\ \zeta_3 & m\zeta_6e^{j\theta}\end{array}\right),$$

$$\mathcal{E}_{16}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}e^{j\alpha} & 0 \\ 0 & e^{j\beta}\end{array}\right)\left(\begin{array}{cc}1 & m\zeta_6e^{j\theta} \\ \zeta_3^2 & m\zeta_3e^{j\theta}\end{array}\right),$$

where $m=\pm1,\pm\zeta_3,m=\pm\zeta_3^2$ and $\theta=\pm\frac{\pi}{3},\pm\frac{2\pi}{3}$.

(2) The second largest gain $|E_2|=\frac{p}{\sqrt{10+2\sqrt{13}}}$ and matrices are given by

$$\mathcal{E}_{21}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}\sqrt{\frac{2}{1+|\lambda_1|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_1|^2}{5+\sqrt{13}}}e^{j\beta}\end{array}\right)\left(\begin{array}{cc}\lambda_1 & 1 \\ -\lambda_1^{-1} & m^2\end{array}\right),$$

$$\mathcal{E}_{22}=\frac{\sqrt{p}}{2}\left(\begin{array}{cc}\sqrt{\frac{2}{1+|\lambda_2|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_2|^2}{5+\sqrt{13}}}e^{j\beta}\end{array}\right)\left(\begin{array}{cc}\lambda_2 & 1 \\ -\zeta_3\lambda_2^{-1} & m^2\end{array}\right),$$

$$\mathcal{E}_{23} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\lambda_3|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_3|^2}{5+\sqrt{13}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \lambda_3 & 1 \\ -\lambda_3^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{24} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\lambda_4|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_4|^2}{5+\sqrt{13}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \lambda_4 & 1 \\ -\zeta_3^2\lambda_4^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{25} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\lambda_5|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_5|^2}{5+\sqrt{13}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \lambda_5 & 1 \\ -\zeta_3\lambda_5^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{26} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\lambda_6|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\lambda_6|^2}{5+\sqrt{13}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \lambda_6 & 1 \\ -\zeta_3^2\lambda_6^{-1} & m^2 \end{pmatrix},$$

where α and β are arbitrary real numbers, $\lambda_1 = \frac{\pm\zeta_3+\sqrt{\zeta_3^2+4}}{2m}$, $\lambda_2 = \frac{\pm1+\sqrt{1+4\zeta_3}}{2m}$, $\lambda_3 = \frac{\pm\zeta_3^2+\sqrt{\zeta_3+4}}{2m}$, $\lambda_4 = \frac{\pm1+\sqrt{1+4\zeta_3^2}}{2m}$, $\lambda_5 = \frac{\pm1+\sqrt{\zeta_3^2+4\zeta_3}}{2m}$ *and* $\lambda_6 = \frac{\pm\zeta_3^2+\sqrt{\zeta_3+4\zeta_3^2}}{2m}$ *with* $m = \pm1, \pm\zeta_3$ *and* $\pm\zeta_3^2$.

(3) The third largest gain is $|E_3| = \frac{p}{\sqrt{10+2\sqrt{21}}}$ *and the matrices are given by*

$$\mathcal{E}_{31} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_1|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_1|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_1 & 1 \\ \mu_1^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{32} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_2|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_2|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_2 & 1 \\ \mu_2^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{33} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_3|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_3|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_3 & 1 \\ \zeta_3\mu_3^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{34} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_4|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_4|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_4 & 1 \\ \zeta_3\mu_4^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{35} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_5|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_5|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_5 & 1 \\ \zeta_3^2\mu_5^{-1} & m^2 \end{pmatrix},$$

$$\mathcal{E}_{36} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu_6|^2}}e^{j\alpha} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu_6|^2}{5+\sqrt{21}}}e^{j\beta} \end{pmatrix}\begin{pmatrix} \mu_6 & 1 \\ \zeta_3^2\mu_6^{-1} & m^2 \end{pmatrix},$$

where α and β are arbitrary real numbers, $\mu_1 = \frac{\pm\zeta_3+\sqrt{\zeta_3^2-4}}{2m}$, $\mu_2 = \frac{\pm\zeta_3^2+\sqrt{\zeta_3-4}}{2m}$, $\mu_3 = \frac{\pm1+\sqrt{1-4\zeta_3}}{2m}$, $\mu_4 = \frac{\pm\zeta_3+\sqrt{\zeta_3^2-4\zeta_3}}{2m}$, $\mu_5 = \frac{\pm1+\sqrt{1-4\zeta_3^2}}{2m}$, $\mu_6 = \frac{\pm\zeta_3^2+\sqrt{\zeta_3-4\zeta_3^2}}{2m}$ *with* $m = \pm1, \pm\zeta_3$ *and* $\pm\zeta_3^2$. ■

Following the idea of Theorem 1 and Theorem 2, we can characterize all two by two full diversity integer generating matrices with the fouth largest gain, the fifth largest gain and so on if necessary. Therefore, we can classify all full diversity integer generating matrices according to their gains.

3. OPTIMAL INTEGER LINEAR SPACE-TIME CODES

In this section, we apply the full diversity integer generating matrices designed in the previous section to a MIMO system with two transmitter antennas and two receiver antennas. We find optimal codes both in Gaussian ring and the Eisentein ring.

Definition 2 *Let* $\mathbf{s} = [s_1, s_2, s_3, s_4]^T \in \mathbb{I}_n^4$. *A codeword matrix* $\mathcal{X}_{\mathbf{G}}(\mathbf{s}) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ *is said to be a norm form integer space-time code if*

$$\begin{pmatrix} x_{11} \\ x_{22} \end{pmatrix} = \mathbf{G}_1 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{12} \\ x_{21} \end{pmatrix} = \mathbf{G}_2 \begin{pmatrix} s_3 \\ s_4 \end{pmatrix},$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are two by two full diversity integer generating matrices with the same power p. ■

Now, thanks to the linearity [19], [20] of the norm form integer space-time codes, our task is to find a pair of full diversity integer generating matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ such that the minimum determinant of $\mathcal{X}_{\mathbf{G}}^H(\mathbf{s})\mathcal{X}_{\mathbf{G}}(\mathbf{s})$ is maximized [21]; i.e.,

$$\max_{\mathrm{tr}(\mathbf{G}_1^H\mathbf{G}_1)=\mathrm{tr}(\mathbf{G}_2^H\mathbf{G}_2)=p} \ \min_{\mathbf{s}\neq\mathbf{0}, \mathbf{s}\in\mathbb{I}_n^4} \det\left(\mathcal{X}_{\mathbf{G}}^H(\mathbf{s})\mathcal{X}_{\mathbf{G}}(\mathbf{s})\right) \tag{7}$$

The following two theorems give the optimal solutions of the above optimization problem.

Theorem 3 *For the Gaussian ring, we have an upper bound,*

$$\min_{\mathbf{s}\neq\mathbf{0}, \mathbf{s}\in\mathbb{I}_n^4} |\det(\mathcal{X}_{\mathbf{G}}(\mathbf{s})| \leq |G_3| = \frac{p}{2\sqrt{5}}$$

with the equality holding when $\mathbf{G}_1 = \mathcal{G}$ *and* $\mathbf{G}_2 = \zeta_8\,\mathcal{G}$, *where*

$$\mathcal{G} = \sqrt{\frac{p}{2}}\begin{pmatrix} \frac{g}{\sqrt{1+g^2}} & \frac{1}{\sqrt{1+g^2}} \\ -\frac{1}{g}\sqrt{\frac{1+g^2}{5}} & \sqrt{\frac{1+g^2}{5}} \end{pmatrix} \tag{8}$$

with the golden number $g = \frac{\sqrt{5}\pm1}{2}$. ■

Dayal and Varanasi [8] proved that their code, the golden code [9]; i.e, code (8), is optimal among all real rotation generating matrices. Our Theorem 3 shows that the golden code is also optimal among all integer generating matrices with the same power.

Theorem 4 *For the Eisentein ring, we have an upper bound,*

$$\min_{\mathbf{s}\neq\mathbf{0}, \mathbf{s}\in\mathbb{I}_n^4} |\det(\mathcal{X}_{\mathbf{G}}(\mathbf{s}))| \leq |E_3| = \frac{p}{\sqrt{10+2\sqrt{21}}}$$

with the equality holding when $\mathbf{G}_1 = \mathcal{E}$ *and* $\mathbf{G}_2 = \zeta_{12}\mathcal{E}$, *where $\mathcal{E}$ is given by*

$$\mathcal{E} = \frac{\sqrt{p}}{2}\begin{pmatrix} \sqrt{\frac{2}{1+|\mu|^2}} & 0 \\ 0 & 2\sqrt{\frac{1+|\mu|^2}{5+\sqrt{21}}} \end{pmatrix}\begin{pmatrix} \mu & 1 \\ \mu^{-1} & \zeta_3 \end{pmatrix}$$

with $\mu = \frac{\zeta_6+\sqrt{\zeta_3+4\zeta_6}}{2}$. ■

If we define the minimum determinant of the infinite code $\mathcal{C}$ [9] as

$$\delta_{\min}(\mathcal{C}) = \min_{\mathcal{X}\in\mathcal{C}, \mathcal{X}\neq\mathbf{0}} |\det(\mathcal{X})|^2 \tag{9}$$

then, Theorem 3 and Theorem 4 tell us that $\delta_{\min}(\mathcal{C}_g) = \frac{p^2}{20}$ and $\delta_{\min}(\mathcal{C}_e) = \frac{p^2}{10+\sqrt{21}}$, where $\mathcal{C}_g$ and $\mathcal{C}_e$ denote all codeword sets generated by $\mathcal{G}$ and $\mathcal{E}$, respectively. Since $\delta_{\min}(\mathcal{C}_g) < \delta_{\min}(\mathcal{C}_e)$, the error performance of the optimal code in the Eisentein ring is better than that of the golden code in the Gaussian ring.

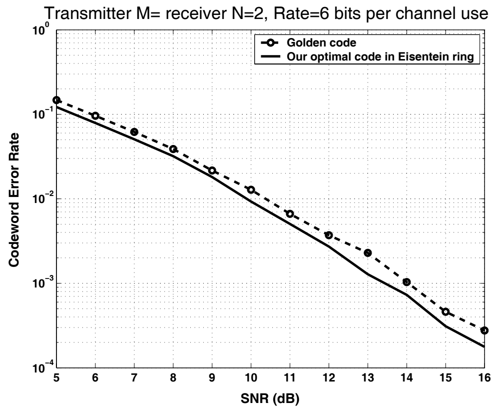

Fig. 1. The performance comparison of the average our codeword error rate with the average error rate of the codewords in [8], [9].

4. SIMULATIONS

In this example, to demonstrate our optimal code given in Theorem 4, we consider a coherent MIMO system with two transmitter antennas and two receiver antennas. Fig. 1 shows that the error performance comparison of our code with those in [8], [9].

5. CONCLUSION

In this paper, we characterized all integer generating matrices for two transmitter antennas. Employing this matrix family to separately design space-time codes layer by layer for two transmitter antenna and two receiver antenna MIMO systems, we proved that the Golden code is optimal among all integer generating matrices with the same power in the Gaussian ring in the sense of maximizing the minimum determinant of codeword matrices. Also, we obtained the optimal code family in the Eisentein ring, the coding gain of which is greater than that of the golden code.

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