ACHIEVING THE WELCH BOUND WITH DIFFERENCE SETS *

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ABSTRACT

Consider a codebook containing N unit-norm complex vectors in a K-dimensional space. In a number of applications, the codebook that minimizes the maximal cross-correlation amplitude ($I_{\max}$) is often desirable. Relying on tools from combinatorial design theory, we construct analytically optimal codebooks meeting, in certain cases, Welch's lower bound. When analytical constructions are not available, we develop an efficient numerical search method based on Lloyd's algorithm, which leads to considerable improvement on the achieved $I_{\max}$ over existing alternatives. We also derive a composite lower bound on the minimum achievable $I_{\max}$ that is effective for any N.

1. INTRODUCTION

Consider a complex (N, K) codebook that is a collection of N unit-norm complex vectors in a K-dimensional vector space. The problem arises often to minimize the codebook's maximal cross-correlation amplitude $I_{\max}$. For multi-antenna transmit beamforming based on limited-rate feedback, minimizing $I_{\max}$ of the beamforming codebook approximately optimizes various performance metrics including average signal-to-noise ratio (SNR), symbol error probability and outage probability [7,10,16]. Minimizing $I_{\max}$ in the context of unitary space time modulations is equivalent to minimizing the block error probability [5]. For multiple description coding over erasure channels, minimizing $I_{\max}$ of the finite frames leads to minimal reconstruction error [13].

Finding the optimal codebook with minimal $I_{\max}$ bears close connection with many other problems in different areas. One equivalent problem is line packing in the Grassmannian space, where one seeks N lines in the K dimensional space so that the maximum chordal distance between any two lines is minimized [2]. In frame theory, such a codebook with $I_{\max}$ minimized is known as a Grassmannian frame [13]. Other closely related problems include the design of equi-angular line sets, spherical t-designs and characterization of strongly regular graphs [13].

Because analytical construction of the optimal codebook is possible only in very special cases [11], numerical search algorithms are often saught to obtain near optimal codebooks. Aside from finding the optimal codebook, lower bounding the achievable $I_{\max}$ is also important. The Welch lower bound [14] on $I_{\max}$ is particularly useful for relatively small values of N; e.g., when $N < K^2$, but becomes quite loose for large N.

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Optimal real codebooks have been studied in [2], while an extensive list of putatively optimal real codebooks can be found in [12]. In this paper, we focus on designing optimal complex codebooks, and our contributions are as follows:

- Relying on tools from combinatorial design theory, we derive analytical constructions of optimal codebooks.
- When analytical constructions do not exist, we employ a numerical search method based on Lloyd's algorithm, which leads to considerable improvement on the achieved $I_{\max}$ of existing designs [5,7].
- We develop a composite lower bound on the achievable $I_{\max}$, that is effective for any N, as verified by our searched codebooks through the Lloyd algorithm.

Notation: Bold upper and lower letters denote matrices and column vectors, respectively; $\mathbf{I}_K$ is the $K \times K$ identity matrix; $(\cdot)^{\mathcal{H}}$ denotes Hermitian transpose; $j = \sqrt{-1}$ is the imaginary unit; $\setminus$ is a set difference operator with $A \setminus B := \{x : x \in A \text{ and } x \notin B\}$; $\mathbb{Z}_+$ denotes the set of positive integers; $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ denotes the set of integers modulo N; $\mathbb{Z}_N^* = \mathbb{Z}_N \setminus \{0\}$; and $\mathbb{F}_N$ is the finite field of order N.

2. PROBLEM FORMULATION AND CONTEXT

Without loss of generality, we consider a codebook comprising N codewords $\mathbf{w}_1, \ldots, \mathbf{w}_N$, with every codeword $\mathbf{w}_\ell$ being a unit norm $K \times 1$ complex vector. We define the $K \times N$ matrix $\mathbf{W} := [\mathbf{w}_1 \ \ldots \ \mathbf{w}_N]$ to represent the (N, K) codebook. The root-mean-square (RMS) cross-correlation and the maximum cross-correlation amplitudes of such a codebook are defined as

$$I_{\rm rms}(\mathbf{W}) := \sqrt{\frac{1}{N(N-1)} \sum_{\ell=1}^{N} \sum_{\ell' \neq \ell}^{N} |\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell'}|^2} \ , \qquad (1)$$

$$I_{\max}(\mathbf{W}) := \max_{1 \le \ell < \ell' \le N} |\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell'}|. \qquad (2)$$

The following results are available [9,14]:

Lemma 1 (Welch's Lower Bound) For any codebook $\mathbf{W}$ with $N \ge K$,

$$I_{\rm rms}(\mathbf{W}) \ge \sqrt{(N-K)/((N-1)K)}, \qquad (3)$$

with equality if and only if $\sum_{\ell=1}^{N} \mathbf{w}_\ell \mathbf{w}_\ell^{\mathcal{H}} = (N/K)\mathbf{I}_K$. Also,

$$I_{\max}(\mathbf{W}) \ge \sqrt{(N-K)/((N-1)K)} \ , \qquad (4)$$

with equality if and only if

$$|\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell'}| = \sqrt{(N-K)/((N-1)K)}, \ \ \forall \ell \neq \ell'. \qquad (5)$$

If equality holds in (3), the codebook $\mathbf{W}$ meets the Welch bound on $I_{\rm rms}$ with equality, and is generally referred to as a WBE codebook. A codebook $\mathbf{W}$ that satisfies (4) as an equality, and thus meets the Welch bound on $I_{\max}$, is referred to as a MWBE codebook. Throughout the paper, the term "Welch bound" denotes the bound on $I_{\max}$ (4), unless explicitly specified. Apparently, if a codebook $\mathbf{W}$ is MWBE, it automatically solves the problem

$$\min_{\mathbf{W}} \max_{\ell \neq \ell'} |\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell'}|, \tag{6}$$

which is of interest in many practical applications.

The following results are also available in the literature:

- An MWBE codebook must be WBE [11], but not vice versa.
- WBE codebooks are "almost trivially easy" to find [11], while minimizing $I_{\max}$ is notoriously difficult in general, both "analytically and numerically" [7, 11].
- Analytic construction of MWBE codebook is very limited. Two known exceptions are:
  i) simplex signalling for an $(N, N-1)$ codebook;
  ii) construction based on conference matrices when $N = 2K = 2^{d+1}$ with $d \in \mathbb{Z}_+$; and when $N = 2K = p^d + 1$ with p a prime number and $d \in \mathbb{Z}_+$ [2].

3. NEW MWBE CODEBOOK CONSTRUCTION

We now present new analytic constructions of complex MWBE codebooks, equipped with tools from combinatorial design theory.

3.1. Systematic construction based on FFT matrices

Noticing that every MWBE codebook must be WBE [11], we can first restrict ourselves to a finite/infinite collection of WBE codebooks, and try to find the desired MWBE codebook (if possible) within this pool of codebooks. Such a strategy would generally simplify the design process, as WBE codebooks are easy to construct and often demonstrate certain favorable structures. We are particularly interested in one special class of WBE codebooks with the following highly restricted structure [5]:

$$\mathbf{W}(\mathbf{u}) = \frac{1}{\sqrt{K}} \begin{bmatrix} 1 & e^{j\frac{2\pi}{N}u_1} & \ldots & e^{j\frac{2\pi}{N}u_1(N-1)} \\ 1 & e^{j\frac{2\pi}{N}u_2} & \ldots & e^{j\frac{2\pi}{N}u_2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{j\frac{2\pi}{N}u_K} & \ldots & e^{j\frac{2\pi}{N}u_K(N-1)} \end{bmatrix}, \tag{7}$$

where $u_k \in \mathbb{Z}_N, \forall 1 \leq k \leq K$ and $u_k \neq u_\ell, \forall k \neq \ell$. Notice that before normalization, $\mathbf{W}(\mathbf{u})$ are nothing but K different rows chosen from an $N \times N$ FFT matrix, while $\mathbf{u} := \{u_1, \ldots, u_K\}$ is the set of selected row indices. Without loss of generality, it is convenient to let every row index range between 0 and $N-1$. Notice that every entry of the codebook is constrained to have the same amplitude $1/\sqrt{K}$. This is desirable in many cases, e.g., equal gain transmit beamforming for multi-antenna wireless communications, and equal gain unitary space time modulations [5]. Interestingly, for every so-constructed codebook the codeword cross-correlation

$$\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell'} = \frac{1}{K} \sum_{k=1}^{K} e^{j\frac{2\pi}{N}u_k(\ell' - \ell)} \tag{8}$$

is determined only by the difference $(\ell' - \ell) \mod N$. Therefore, the codebook $\mathbf{W}(\mathbf{u})$ has a circulant correlation structure [5].

The codebook $\mathbf{W}(\mathbf{u})$ is now determined by only K parameters. Choice of the row indices $\mathbf{u}$ vastly influences $I_{\max}$ of the so-constructed codebook; see [5, Figs. 2 and 3] for an illustrating example. Theoretically, one can perform an exhaustive search over all possible $\binom{N}{K}$ row indices, in an attempt to find a MWBE codebook from this pool of codebooks. However, as $\binom{N}{K}$ increases explosively with N, a randomly incomplete search is often employed instead of the exhaustive search. This is precisely the codebook search algorithm developed in [5].

The WBE codebook in (7) may not produce MWBE codebooks, because of the strict constraints. We show next that such a highly structured approach still enables analytic MWBE codebook construction for certain (N, K) pairs. The core questions are then: when can we construct MWBE codebooks with structure as in (7)? and how can we achieve this without numerical search?

3.2. Codebook constructions based on difference sets

Based on the necessary and sufficient conditions in (5), the MWBE codebook design problem now boils down to

$$\text{find } \mathbf{u}, \text{ s. t. } f(1) = \ldots = f(N-1) = \sqrt{\frac{N-K}{(N-1)K}}, \tag{9}$$

where

$$f(m) := |\mathbf{w}_\ell^{\mathcal{H}} \mathbf{w}_{\ell-m}| = \frac{1}{K} \left| \sum_{k=1}^{K} e^{j2\pi\frac{m}{N}u_k} \right|, \tag{10}$$

with $f(0) \equiv 1$ and $f(m) \equiv f(N-m)$.

We start by investigating two trivial cases: $K = N$ and $K = N-1$. For $K = N$, selecting all the N rows from the FFT matrix forms a trivial (N, N) MWBE codebook; and therefore, $\mathbf{u} = \mathbb{Z}_N$ is an optimal solution to (9) when $K = N$.

When $K = N-1$, let us select arbitrary $N-1$ rows from the FFT matrix by excluding the ℓth row. It directly follows that $\forall m \in \mathbb{Z}_N^*$, and $\forall \ell \in \mathbb{Z}_N$,

$$f(m) = \frac{1}{K} \left| \sum_{k=0, k\neq\ell}^{N-1} e^{j\frac{2\pi}{N}km} \right| = \frac{1}{K} = \sqrt{\frac{N-K}{(N-1)K}}. \tag{11}$$

Therefore, choosing arbitrary $N-1$ rows from the FFT matrix forms a $(N, N-1)$ MWBE codebook, which corresponds to a simplex design of N signals in an $N-1$ dimensional space.

We next present a non-trivial case with the proof skipped due to space limitations.

Lemma 2: The codebook $\mathbf{W}(\mathbf{u})$ is MWBE if $\mathbf{u} = \{t^2 \mod N : t \in \mathbb{Z}_N^*\}$ and $N = 3 \mod 4$ is a prime.

We find that the optimal row indices $\mathbf{u}$ in the aforementioned three different cases, trivial or non-trivial, share a subtle resemblance.

Definition: A subset $\mathbf{u} = \{u_1, \ldots, u_K\}$ of $\mathbb{Z}_N$ is called a (N, K, λ) *difference set* if the $K(K-1)$ differences $(u_k - u_\ell) \mod N, k \neq \ell$, take all possible nonzero values $1, 2, \ldots, N-1$, with each value exactly λ times.

It can be readily verified that in the above three special cases, $\mathbf{u} = \mathbb{Z}_N$ is a (N, N, N) difference set, $\mathbf{u} = \mathbb{Z}_N^*$ is a $(N, N-1, N-2)$ difference set, while $\mathbf{u} = \{t^2 \mod N : t \in \mathbb{Z}_N^*\}$ is a $(N, (N-1)/2, (N-3)/4)$ difference set when $N = 3 \mod 4$ is a prime. Notice that the three parameters (N, K, λ) are not independent by definition, and $\lambda(N-1) \equiv K(K-1)$. We are thus motivated to derive the following general results.

Theorem 1: The (N,K) codebook $\mathbf{W}(\mathbf{u})$ is MWBE if $\mathbf{u}$ is a (N,K,λ) difference set.

Proof: For any $m \in \mathbb{Z}_N^* = \{1,2,\ldots,N-1\}$, we have

$$\begin{aligned} f_m^2 &= \frac{1}{K^2}\left(\sum_{i=1}^{K} e^{j2\pi m u_i/N}\right)\left(\sum_{k=1}^{K} e^{-j2\pi m u_k/N}\right) \\ &= \frac{1}{K^2}\sum_{i=1}^{K}\sum_{k=1}^{K} e^{j2\pi m(u_i-u_k)/N} \\ &= \frac{K}{K^2} + \frac{\lambda \sum_{\ell=1}^{N-1} e^{j2\pi m\ell/N}}{K^2} \\ &= \frac{K-\lambda}{K^2} = \frac{N-K}{(N-1)K}\,, \end{aligned}$$

where the fourth equality follows from the definition of the difference set, and the fifth equality is due to the equalities $\sum_{\ell=0}^{N-1} e^{j2\pi m\ell/N} = 0, \forall m \in \mathbb{Z}_N^*$. The necessary and sufficient conditions in (5), establish that $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook. □

Difference sets have been well studied in the combinatorial design theory [1, 4] and are known to exist for certain pairs of parameters (N,K), while the search for new difference sets is still under way. We list in the following several families of non-trivial MWBE codebook designs based on difference sets [1, 4]. For illustration purpose, we also tabulate several non-trivial codebook examples in Table I.

Family 1 — MWBE codebooks based on Singer difference sets: Let $q = p^r$ be a power of a prime, $d \geq 2$ be a positive integer, α be a generator of the multiplicative group of $\mathbb{F}_{q^{d+1}}$,

$$trace(\omega) = \sum_{i=0}^{d} \omega^{q^i}$$

be the trace function from $\mathbb{F}_{q^{d+1}}$ to $\mathbb{F}_q$, and

$$N = \frac{q^{d+1}-1}{q-1},\ K = \frac{q^d-1}{q-1},\ \lambda = \frac{q^{d-1}-1}{q-1}. \qquad (12)$$

Then $\mathbf{u} = \{t : 0 \leq t < N, trace(\alpha^t) = 0\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook.

Family 2 — MWBE codebooks based on Quadratic difference sets: Let $q = p^r = 3 \mod 4$ be a power of a prime and $N = q, K = (q-1)/2, \lambda = (q-3)/4$. Then $\mathbf{u} = \{t^2 : t \in \mathbb{Z}_N^*\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook. Lemma 2 is a special case of this family when $r = 1$.

Family 3 — MWBE codebooks based on Quartic difference sets: Let $p = 4a^2+1$ be a prime with a odd, and $N = p, K = (p-1)/4, \lambda = (p-5)/16$. Then $\mathbf{u} = \{t^4 : t \in \mathbb{Z}_N^*\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook. Similarly, let $p = 4a^2+9$ be a prime with a odd, and $N = p, K = (p+3)/4, \lambda = (p+3)/16$. Then $\mathbf{u} = \{t^4 : t \in \mathbb{Z}_N\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook.

Family 4 — MWBE codebooks based on Octic difference sets: Let $p = 8a^2+1 = 64b^2+9$ be a prime with a,b odd, and $N = p, K = a^2, \lambda = b^2$. Then $\mathbf{u} = \{t^8 : t \in \mathbb{Z}_N^*\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook. Similarly, let $p = 8a^2+49 = 64b^2+441$ be a prime with a odd, b even, and $N = p, K = a^2+6, \lambda = b^2+7$. Then $\mathbf{u} = \{t^4 : t \in \mathbb{Z}_N\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook.

Family 5 — MWBE codebooks based on Twin-primes difference sets: Let p and $q = p+2$ be a pair of twin primes, g be a common primitive root of both p and q, and $N = pq, K = (pq-1)/2, \lambda = (pq-3)/4$. Then $\mathbf{u} = \{1, g, g^2 \mod N, \ldots, g^{(p^2-3)/2} \mod N, 0, q, \ldots, (p-1)q\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook. Similarly, let p,q be a pair of twin primes such that $(p-1, q-1) = 4$ with $d = (p-1)(q-1)/4$, g be a common primitive root of both p and q, and $N = pq, K = (pq-1)/4, \lambda = (pq-5)/16$. Then $\mathbf{u} = \{1, g, g^2 \mod N, \ldots, g^{d-1} \mod N, 0, q, 2q, \ldots, (p-1)q\}$ is a (N,K,λ) difference set, and $\mathbf{W}(\mathbf{u})$ forms a (N,K) MWBE codebook.

Notice that for every (N,K,λ) difference set $\mathbf{u} \in \mathbb{Z}_N$, the complement $\bar{\mathbf{u}} = \mathbb{Z}_N \setminus \mathbf{u}$ is a $(N, N-K, N-2K+\lambda)$ difference set [1]. To the best of our knowledge, only the special case of $d = 2$ in family 1 has been separately reported in the pursuit of equiangular vector sets [6], while the general concept of difference sets has not been recognized therein.

N	K	FFT row indices $\mathbf{u}$	Family [1, 4]
7	3	{1, 2, 4}	Quadratic
13	4	{0, 1, 3, 9}	Singer, $d = 2$
11	5	{1, 3, 4, 5, 9}	Quadratic
31	6	{1, 5, 11, 24, 25, 27}	Singer, $d = 2$
15	7	{0, 1, 2, 4, 5, 8, 10}	Twin-primes
37	9	{1, 7, 9, 10, 12, 16, 26, 33, 34}	Quartic
73	9	{1, 2, 4, 8, 16, 32, 37, 55, 64}	Octic

Table I. Some optimal codebook examples.

4. LLOYD SEARCH ALGORITHM AND COMPOSITE LOWER BOUND ON $I_{\max}$

4.1. Numerical Search based on the Lloyd algorithm

When analytic codebook construction is not possible, computationally complex numerical search has to be employed to find a near optimal solution to (6). A brute-force computer search is used in [7], by choosing the codebook with the smallest $I_{\max}$ from a large group of randomly generated codebooks. Alternatively, one can perform an exhaustive, or incomplete, search for MWBE codebook from a finite/infinite collection of WBE codebooks, as in [5].

Seeking a more efficient numerical search, we consider a vector quantizer, where the N codeword vectors $\mathbf{w}_1, \ldots, \mathbf{w}_N$ and the random source input $\mathbf{g}$ are all $K \times 1$ complex vectors constrained on the complex unit hypersphere Ω^K; and in particular, $\mathbf{g}$ is uniformly distributed on Ω^K (see [15] for extensions to correlated channels). Suppose the distortion metric between $\mathbf{w}_\ell$ and $\mathbf{g}$ is defined as $d(\mathbf{g}, \mathbf{w}_\ell) := 1 - |\mathbf{w}_\ell^{\mathcal{H}}\mathbf{g}|^2$. It has been shown in [7, 10] that the optimal quantizer codebook minimizing the overall average distortion serves as a good candidate toward minimizing $I_{\max}$. We therefore take as candidates all interim quantizer codebooks at the output of every Lloyd iteration. The one with smallest $I_{\max}$ should provide a near optimal solution to (6). To further bring down $I_{\max}$, the Lloyd algorithm can be carried out many times with randomly different initializations.

For several pairs of N and K, Table II lists the minimal $I_{\max}$ obtained by different search algorithms, while results of the brute-force search method are taken directly from [8]. As we can see, the Lloyd search algorithm generally produces better codebooks with considerably smaller $I_{\max}$ (codebooks available in authors' homepage).

N	K	Lloyd based	[8]	[5]	Welch bound
8	2	0.82161	0.84152	0.92388	0.65465
16	3	0.67658	0.80793	0.78973	0.53748
16	4	0.45139	0.75252	0.58171	0.44721
64	4	0.74468	0.79731	0.79731	0.48795

Table II. Comparison of numerical search results

4.2. Composite lower bound on $I_{\max}$

The Welch lower bound on $I_{\max}$ is very useful when $N < K^2$, but becomes quite loose for large N. We are therefore motivated to look for a tighter bound on $I_{\max}$ in such cases.

Theorem 2: For a (N, K) codebook, $I_{\max}$ is lower bounded by

$$I_{\max} \geq \max\left(\sqrt{\frac{N-K}{(N-1)K}},\ 1-2N^{-\frac{1}{K-1}}\right). \qquad (13)$$

Proof: Apparently, half of this composite bound is due to the Welch bound. The other half

$$I_{\max} \geq 1-2N^{-\frac{1}{K-1}} \qquad (14)$$

is mainly due to [10], although not explicitly stated as a bound on $I_{\max}$ therein. In fact, by plugging [10, Eq. (58)] into the second equality of [10, Eq. (55)] and realizing the fact that any probability is always less than or equal to 1, we can easily obtain (14). □

Based on results from [10], another bound becomes available [7, Theorem 2]:

$$I_{\max} \geq \left(1-4N^{-1/(K-1)}\right)^{1/2}, \qquad (15)$$

which is always inferior to the new bound in (14). For $K = 2$ and different values of N, Fig. 1 plots various bounds on $I_{\max}$ together with the minimum achieved $I_{\max}$ through analytic construction or by using the Lloyd search algorithm. As N increases, the Welch bound becomes increasingly looser while the new bound (14) gets increasingly tighter. The composite bound in (13) is thus effective throughout the range of N. This figure also demonstrates the efficiency of Lloyds' search algorithm, as most codebooks obtained by numerical search yield $I_{\max}$ close to the bound. Also notice that bound (15) is always below bound (14).

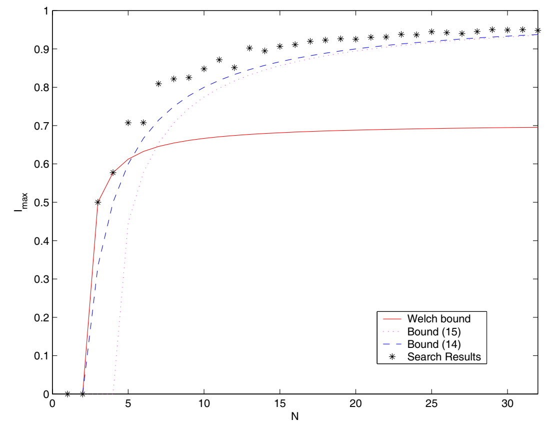

Fig. 1. Lower bounds and minimum achieved values of $I_{\max}$.

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