REDUCING THE AVERAGE COMPLEXITY OF ML DETECTION USING SEMIDEFINITE RELAXATION

*Joakim Jaldén**, *Björn Ottersten*

Dept. Signal, Sensors & Systems,
Royal Institute of Technology (KTH)
Stockholm, Sweden

Wing-Kin Ma

Dept. Electrical & Electronic Engineering,
University of Melbourne
Parkville, Vic., Australia

ABSTRACT

Maximum likelihood (ML) detection of symbols transmitted over a MIMO channel is generally a difficult problem due to its NP-hard nature. However, not every instance of the detection problem is equally hard. Thus, the average complexity of an ML detector may be significantly smaller than its worst-case counterpart. This is typically true in the high SNR regime where the received signals are closer to the noise free transmitted signals. Herein, a method which may be used to lower the average complexity of any ML detector is proposed. The method is based on the ability to verify if a symbol estimate is ML, using an optimality condition provided by the near-ML semidefinite relaxation technique. The average complexity reduction advantage of the proposed method is confirmed by numerical results.

1. INTRODUCTION

Maximum Likelihood (ML) detection in multiple-input multiple-output (MIMO) channels is a problem that requires joint detection of multiple symbols at the receiver. For a general channel where no special exploitable structure exists, the ML detection problem has been shown to be NP-hard [1, 2]. This implies that there is no known algorithm which solves an arbitrary instance of the problem in polynomial time. However, in digital communications the instances of the ML detection problem are generated according to some stochastic model due to the random behavior of the communication channel and it is therefore possible to have a scenario where most instances are relatively easy to solve. An algorithm which is able to exploit this property may have a low average complexity or efficiently solve the detection problem with a high probability.

The above mentioned complexity behavior has been widely recognized in the now popular sphere decoding algorithm [3, 4] which can be used to solve the ML detection problem. Although sphere decoding yields an exponential expected complexity [5] with respect to the problem size, it enjoys fairly low average complexity in the high SNR regime and for problems of moderate size [4]. This is because the sphere decoding complexity is dependent on the distance between the received message and the noise free transmitted message. As the SNR is increased, i.e. the noise variance decreased, this distance becomes smaller on average thereby leading to a decrease in the average complexity.

A natural question is whether such simplifying properties of the detection problem can be exploited in other implementation or approximations of the ML detector as well. Herein, an add-on procedure which is applicable to any ML detection algorithm is proposed. This method is based on the semidefinite relaxation (SDR) approach previously proposed in [6, 7] and is applicable when binary constellations such as BPSK or QPSK are used. The optimization is based on the observation that in certain instances it is possible to verify whether a particular estimate of the transmitted message is the true ML estimate. The key idea is to first apply a computationally efficient detector, such as the zero-forcing (ZF) detector to estimate the transmitted message. Then, by the application of the SDR framework it is tested if this estimate corresponds to the true ML estimate. If this is the case, there is no need to actually run the presumably more complex ML detector. Additionally, if this ML test is affirmative with high probability the ML detection algorithm needs only to be applied to a small number of problem instances, thereby reducing the average complexity.

The outline of this paper is as follows. In Section 2 the channel model and ML detector are presented. Then the principle of semidefinite relaxation is treated in Section 3. The proposed ML optimization is outlined in Section 4. The efficiency of the method, i.e. the probability that the test is positive, is then addressed by numerical simulations in Section 5.

2. ML DETECTION

Consider a discrete time linear MIMO channel modeled on matrix form as

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{v}. \tag{1}$$

The vector $\mathbf{y} \in \mathbb{R}^n$ is the received signal, the matrix $\mathbf{H} \in \mathbb{R}^{n\times m}$ is the channel matrix, and the vector $\mathbf{v} \in \mathbb{R}^n$ is additive white Gaussian noise. The transmitted symbols, $\mathbf{s}$, are drawn from a BPSK constellation, i.e. $\mathbf{s} \in \mathcal{B}^m$ where $\mathcal{B} \triangleq \{\pm 1\}$. It will also herein be assumed that $n \geq m$. Also, $\mathbf{s}$ will denote the vector of symbols actually transmitted across the channel. Estimates of $\mathbf{s}$ will be denoted by $\hat{\mathbf{s}}$ and arbitrary vectors in $\mathcal{B}^m$ such as optimzation variables will be denoted by $\bar{\mathbf{s}}$.

The channel model is written in real valued form simply because this is convenient in the SDR framework. It is however well known that the complex valued model with a QPSK constellation may be written in the form of (1) by rewriting the original model as

$$\begin{bmatrix}\Re(\mathbf{y})\\ \Im(\mathbf{y})\end{bmatrix} = \begin{bmatrix}\Re(\mathbf{H}) & -\Im(\mathbf{H})\\ \Im(\mathbf{H}) & \Re(\mathbf{H})\end{bmatrix}\begin{bmatrix}\Re(\mathbf{s})\\ \Im(\mathbf{s})\end{bmatrix} + \begin{bmatrix}\Re(\mathbf{v})\\ \Im(\mathbf{v})\end{bmatrix} \tag{2}$$

where $\Re$ and $\Im$ denote the real and imaginary parts respectively.

*jalden@s3.kth.se

Under the white Gaussian noise assumption the ML detector of $\mathbf{s}$, $\hat{\mathbf{s}}_{\text{ML}}$, is given by

$$\hat{\mathbf{s}}_{\text{ML}} \triangleq \underset{\bar{\mathbf{s}}\in\mathcal{B}^m}{\operatorname{argmin}} \|\mathbf{y}-\mathbf{H}\bar{\mathbf{s}}\|^2, \tag{3}$$

i.e. among all hypothesized messages, $\mathbf{H}\bar{\mathbf{s}}$, the one which yields the smallest distance to $\mathbf{y}$ is chosen. Problem (3) is however NP-hard [1, 2] and solving (3) by exhaustive search has a complexity which grows as 2^m. This makes computationally less complex solutions of (3) interesting.

3. SEMIDEFINITE RELAXATION

The ML detector in (3) can be equivalently written as

$$\hat{\mathbf{s}}_{\text{ML}} = \underset{\bar{\mathbf{s}}\in\mathcal{B}^m}{\operatorname{argmin}}\ \bar{\mathbf{s}}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{H}\bar{\mathbf{s}} - 2\mathbf{y}^{\mathrm{T}}\mathbf{H}\bar{\mathbf{s}}.$$

By noting that $\mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x} = \operatorname{Tr}\left(\mathbf{L}\mathbf{x}\mathbf{x}^{\mathrm{T}}\right)$ and letting $\mathbf{x} = [\bar{\mathbf{s}}^{\mathrm{T}}\ 1]^{\mathrm{T}}$, it can be shown [7] that the above problem is equivalent to

$$\begin{array}{ll} \underset{\mathbf{X},\,\mathbf{x}}{\min} & \operatorname{tr}(\mathbf{L}\mathbf{X}) \\ \text{s.t.} & \operatorname{diag}(\mathbf{X}) = \mathbf{e} \\ & \mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}} \end{array} \tag{4}$$

where $\mathbf{e}$ is the vector of all ones,

$$\mathbf{L} = \begin{bmatrix} \mathbf{H}^{\mathrm{T}}\mathbf{H} & -\mathbf{H}^{\mathrm{T}}\mathbf{y} \\ -\mathbf{y}^{\mathrm{T}}\mathbf{H} & 0 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} \bar{\mathbf{s}} \\ 1 \end{bmatrix}.$$

Note that the constraint $\operatorname{Diag}(\mathbf{x}\mathbf{x}^{\mathrm{T}}) = \mathbf{e}$ ensures that $\bar{\mathbf{s}} \in \mathcal{B}^m$. Also, $[\mathbf{x}]_{m+1} = 1$ where $[\mathbf{x}]_i$ is ith component of $\mathbf{x}$ does not need to be maintained explicitly since if $\mathbf{x}$ is a solution to (4) then so is it's negative value, $-\mathbf{x}$.

Problem (4) is equivalent to (3) in the sense that the solution of one is easily obtained from the solution of the other. This implies that (4) is also NP-hard. However, if the constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ is replaced by $\mathbf{X} \succeq \mathbf{0}$ where $\mathbf{X} \succeq \mathbf{0}$ means that $\mathbf{X}$ is symmetric and positive semidefinite (PSD), problem (4) becomes

$$\begin{array}{ll} \underset{\mathbf{X}}{\min} & \operatorname{tr}(\mathbf{L}\mathbf{X}) \\ \text{s.t.} & \operatorname{diag}(\mathbf{X}) = \mathbf{e} \\ & \mathbf{X} \succeq \mathbf{0} \end{array} \tag{5}$$

which is a convex optimization problem that can be solved efficiently in $\mathcal{O}(m^{3.5})$ time [8]. Since $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ implies that $\mathbf{X} \succeq \mathbf{0}$, problem (5) represents a relaxation of (4) and the optimal objective value of (5) gives a lower bound on that of (4). Techniques where the solution to (5) is used to approximate the solution of (4) have previously been used in the communications literature to obtain estimators, $\hat{\mathbf{s}}$, which have near ML performance [6, 7]. Here, our interest is not in numerical optimization of (5), which has been considered elsewhere [8, 6]. We are interested in some properties of (5) that will prove useful in complexity reduction for a generic ML detector.

4. ML VERIFICATION

Note that if for some reason the solution of (5), $\mathbf{X}$, were to be of rank one then it could be factored as $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ and would also solve (4) and (3). In [9], the necessary and sufficient condition for (5) to have a rank one solution corresponding to $\bar{\mathbf{s}}$ was obtained. This result, proven in [9], is given by Theorem 1 below.

Theorem 1 *Let $\bar{\mathbf{s}}$ be any vector in $\mathcal{B}^m$. Also let $\bar{\mathbf{v}} \triangleq \mathbf{y} - \mathbf{H}\bar{\mathbf{s}}$ and define a set*

$$\mathcal{V}_{\bar{\mathbf{s}}} \triangleq \{\bar{\mathbf{v}} \mid \mathbf{H}^{\mathrm{T}}\mathbf{H} + \operatorname{Diag}(\bar{\mathbf{s}})^{-1}\operatorname{Diag}(\mathbf{H}^{\mathrm{T}}\bar{\mathbf{v}}) \succeq \mathbf{0}\}.$$

Then, $\bar{\mathbf{v}} \in \mathcal{V}_{\bar{\mathbf{s}}}$ if and only if

$$\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}, \quad \mathbf{x} = [\bar{\mathbf{s}}^{\mathrm{T}} 1]^{\mathrm{T}}$$

is a solution to (5).

The following corollary is a direct consequence of the theorem since any rank one solution of (5) is also a solution of (4).

Corollary 2 *Let $\hat{\mathbf{s}} \in \mathcal{B}^m$ be any estimate of $\mathbf{s}$. Let $\hat{\mathbf{v}}$ and $\mathcal{V}_{\hat{\mathbf{s}}}$ be given as in Theorem 1. Then*

$$\hat{\mathbf{v}} \in \mathcal{V}_{\hat{\mathbf{s}}} \quad \Rightarrow \quad \hat{\mathbf{s}} = \hat{\mathbf{s}}_{\mathrm{ML}}.$$

Note the implication in the corollary is only in one direction. It is possible that $\mathbf{s}_{\text{ML}} = \hat{\mathbf{s}}$ while $\hat{\mathbf{v}} \notin \mathcal{V}_{\hat{\mathbf{s}}}$. This is a consequence of that the SDR is not guaranteed to always yield rank one solutions.

However, what is stated by Corollary 2 is that if an estimate, $\hat{\mathbf{s}}$, is obtained by some estimation procedure and it turns out that $\hat{\mathbf{v}} \in \mathcal{V}_{\hat{\mathbf{s}}}$ then it is known that $\hat{\mathbf{s}}$ is the same estimate as would have been obtained by the ML detector. This observation allows for the following potential average complexity reduction of any ML detector.

1. Obtain an estimate $\hat{\mathbf{s}}$ by some computationally efficient detection method, e.g. by the zero-forcing detector as

$$\hat{\mathbf{s}} = \sigma(\mathbf{H}^{\dagger}\mathbf{y}) \tag{6}$$

   where $\sigma(\cdot)$ is the sign function and † is the Moore-Penrose pseudo inverse [10].

2. Compute $\hat{\mathbf{v}} = \mathbf{y} - \mathbf{H}\hat{\mathbf{s}}$ and test

$$\mathbf{Q} \triangleq \mathbf{H}^{\mathrm{T}}\mathbf{H} + \operatorname{Diag}(\hat{\mathbf{s}})^{-1}\operatorname{Diag}(\mathbf{H}^{\mathrm{T}}\hat{\mathbf{v}}) \succeq \mathbf{0}. \tag{7}$$

   If positive, declare $\hat{\mathbf{s}}_{\text{ML}} = \hat{\mathbf{s}}$. Otherwise, solve or approximate (3) by some method of choice.

Note that the test in (7) can be done by applying Cholesky factorization, the operational cost of which is $\frac{1}{3}m^3$. Now, assume that (7) is true with high probability and the algorithm chosen to solve (3) has a computational complexity which is higher than that required for testing (7). Then the above strategy will lower the average computational complexity of obtaining $\hat{\mathbf{s}}_{\mathrm{ML}}$ since the ML detector will only need to be used for a fraction of the problem instances. That there are scenarios for which (7) is indeed true with high probability is shown by the numerical examples in Section 5.

There are also analytical arguments that indicate scenarios under which the above scheme may prove useful. Note that the probability of (7) being true tends to one with increasing SNR for any given full rank channel matrix, $\mathbf{H}$. This follows since $\mathbf{0}$ is in the interior of $\mathcal{V}_{\mathbf{s}}$ and the norm of $\mathbf{v} = \mathbf{y} - \mathbf{H}\mathbf{s}$ is small with high probability in the high SNR range. It can also be shown that (7) is always true if the channel matrix, $\mathbf{H}$, has orthogonal columns, see [9]. It is therefore reasonable to assume that the proposed method will work well when the SNR is high and the channel matrix is relatively well conditioned. This is in line with the intuition that the ML detection problem is not as difficult for such instances.

4.1. Partial Factorizations

Even in the case where (7) is not satisfied, some additional information about the estimate, $\hat{\mathbf{s}}$, may still be obtained by a partial Cholesky factorization of $\mathbf{Q}$. The partial Cholesky factor of $\mathbf{Q}$ is a side product of attempting to find the complete Cholesky factor [11], and hence obtaining the partial Cholesky factor does not incur complexity increase of the proposed method. Information obtained from the partial Cholesky factorization may however be useful in a scheme where symbols are detected sequentially, such as decision feedback detectors, block detectors and sphere decoders.

Assume that $\mathbf{Q} \not\succeq \mathbf{0}$ which causes the Cholesky factorization algorithm[1] to terminate early after having processed row and column k of $\mathbf{Q}$. Then, partition $\mathbf{Q}$ as

$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$$

where $\mathbf{Q}_{11} \in \mathbb{R}^{k \times k}$ has been confirmed by the algorithm to be PSD, i.e. $\mathbf{Q}_{11} \succeq \mathbf{0}$. Also, k is the largest integer such that $\mathbf{Q}_{11} \succeq \mathbf{0}$ holds true. Now, let

$$\mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix}$$

where $\mathbf{H}_1$ contains the first k columns of $\mathbf{H}$. Similarly let $\hat{\mathbf{s}} = [\hat{\mathbf{s}}_1^{\mathrm{T}}\ \hat{\mathbf{s}}_2^{\mathrm{T}}]^{\mathrm{T}}$. Then, $\mathbf{Q}_{11}$ can be written as

$$\mathbf{Q}_{11} = \mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1 + \mathrm{Diag}(\hat{\mathbf{s}}_1)^{-1}\mathrm{Diag}(\mathbf{H}_1^{\mathrm{T}}\hat{\mathbf{v}}).$$

Since $\mathbf{Q}_{11} \succeq \mathbf{0}$ by assumption and

$$\hat{\mathbf{v}} = \mathbf{y} - \mathbf{H}_1\hat{\mathbf{s}}_1 - \mathbf{H}_2\hat{\mathbf{s}}_2 = \bar{\mathbf{y}} - \mathbf{H}_1\hat{\mathbf{s}}_1$$

where

$$\bar{\mathbf{y}} \triangleq \mathbf{y} - \mathbf{H}_2\hat{\mathbf{s}}_2,$$

it follows that

$$\hat{\mathbf{s}}_1 = \underset{\bar{\mathbf{s}}_1 \in \mathcal{B}^k}{\mathrm{argmin}} \|\bar{\mathbf{y}} - \mathbf{H}_1\bar{\mathbf{s}}_1\|^2.$$

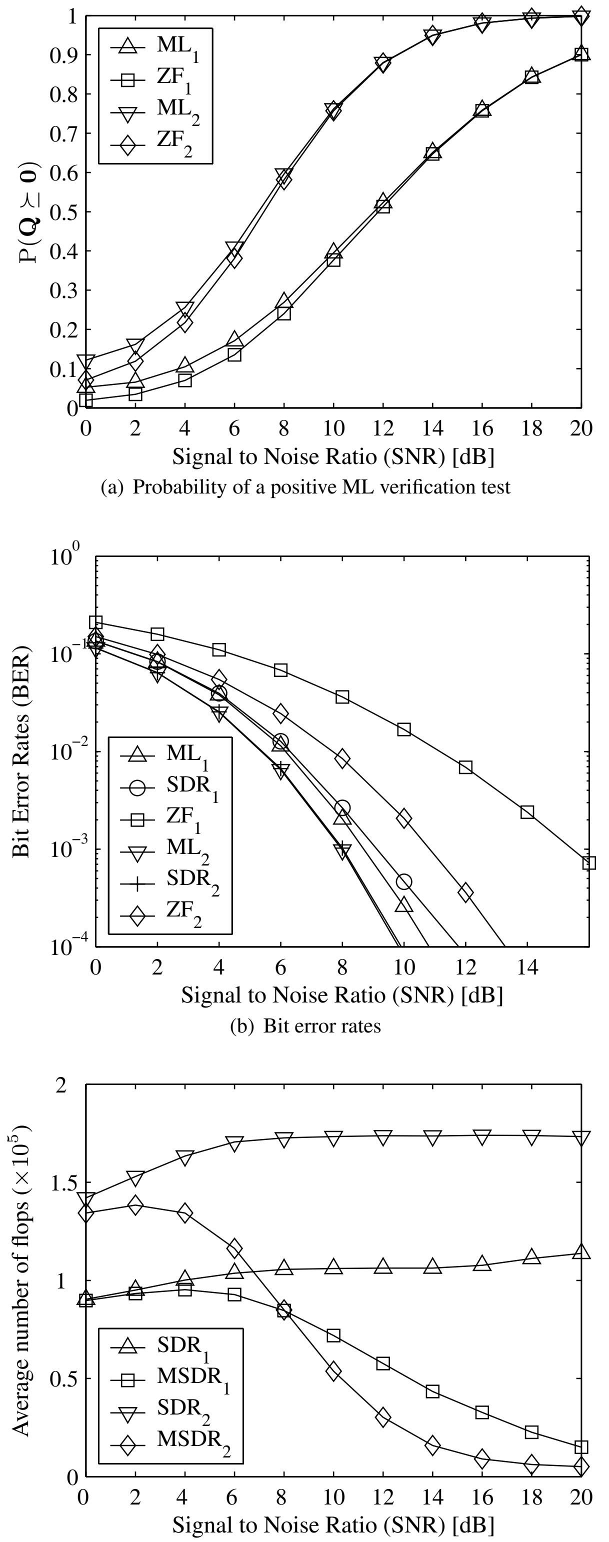
That is, even though $\hat{\mathbf{s}} = [\hat{\mathbf{s}}_1^{\mathrm{T}}\ \hat{\mathbf{s}}_2^{\mathrm{T}}]^{\mathrm{T}}$ does not equal the ML estimate of $\mathbf{s}$ it is known that $\hat{\mathbf{s}}$ is the conditional ML estimate given that the last $m - k$ components of $\mathbf{s}$ are equal to $\hat{\mathbf{s}}_2$.

Therefore, if the estimate $\hat{\mathbf{s}}$ is to be further improved to yield a lower objective value in (3) the part given by $\hat{\mathbf{s}}_2$ must be changed. This could potentially be useful in, for instance, a sphere decoder algorithm [3]: if $\hat{\mathbf{s}}$ were any vector found in the search sphere, it is known that the algorithm could safely retrace to symbol $[\hat{\mathbf{s}}]_{k+1}$ before continuing the search for the ML estimate. Additionally, as previously outlined, if $k = m$ the sphere decoder algorithm could be terminated and $\hat{\mathbf{s}} = \hat{\mathbf{s}}_{\mathrm{ML}}$ declared.

5. SIMULATIONS

The performance of the proposed method is clearly dependent on the probability that $\mathbf{Q} \succeq \mathbf{0}$. The probability directly establishes the fraction of problem instances for which the ML detector needs to be applied. It will typically depend on the particular estimate, $\hat{\mathbf{s}}$, used in (7) as well as the statistics of the noise, $\mathbf{v}$, and the channel matrix, $\mathbf{H}$. While an analytical evaluation of this probability seems intractable it may be evaluated numerically by Monte Carlo simulations which is done in this section.

[1] It is herein assumed that this algorithm is implemented as in [11].



(a) Probability of a positive ML verification test

(b) Bit error rates

(c) Computational complexity of original SDR and modified SDR

Fig. 1. Performance of the ML verification procedure. In scenario 1 $(N, M) = (8, 6)$ and in scenario 2 $(N, M) = (12, 6)$. The scenario number is indicated by the subscript.

Instances of the ML detection problem were generated according to the i.i.d. Rayleigh fading multiple antenna model, i.e. the entries of $\mathbf{H} \in \mathbb{C}^{N \times M}$ were generated independently according to a complex Gaussian distribution. The signal to noise ratio (SNR) was defined as

$$\mathrm{SNR} \triangleq \frac{\mathrm{E}\{\|\mathbf{Hs}\|^2\}}{\mathrm{E}\{\|\mathbf{v_H}\|^2\}}$$

where $\mathbf{v_H}$ is the projection of the noise, $\mathbf{v}$, onto the range of $\mathbf{H}$, i.e. the part of the noise relevant to the detection.

To be able to study the dependence on the conditioning of the channel matrix, we consider the following two channel scenarios: $(N, M) = (8, 6)$, and $(N, M) = (12, 6)$. On average the channel matrix of the second scenario is better conditioned. In both cases, according to (2), the number of BSPK symbols detected, m, is 12. Throughout the numerical study we use a sphere decoder to obtain the ML estimates which are used for the performance comparisons.

In Fig. 1(a) the probability of $\mathbf{Q}$ being positive semidefinite is shown as a function of the SNR. The initial estimate, $\hat{\mathbf{s}}$, in (7) was chosen as the ZF estimate given by (6). As expected, the probability of $\mathbf{Q} \succeq \mathbf{0}$ increases with the SNR. As a reference, in Fig. 1(a) we include the probability of $\mathbf{Q} \succeq \mathbf{0}$ when the initial estimate in (7) is the true ML estimate. This optimal initialization is useless in practice, but from a performance study viewpoint it gives us an upper bound on the possible performance of the proposed method which could be attained by choosing $\hat{\mathbf{s}}$ appropriately. By the closeness of the curves in Fig. 1(a) it can be seen that not much will be gained in these scenarios by considering more advanced estimators than the ZF. Furthermore, the degradation of the proposed method when the channel is poorly conditioned can be seen in the figure.

Fig. 1(b) shows the performance of the ML, SDR and ZF detectors in the two scenarios. This figure is included mainly for reference and the performance of the detectors has been thoroughly investigated elsewhere, see e.g. [7]. By comparing Fig. 1(b) and 1(a) it is seen that the probability of Corollary 2 being unable to identify an ML solution is more significant than the probability that the ZF estimate, $\hat{\mathbf{s}}$, is not equal to the ML estimate. Thus, the proposed method is limited by the effectiveness of (7) rather than the accuracy of $\hat{\mathbf{s}}$. Still, in the high SNR regime a large portion of the estimates, $\hat{\mathbf{s}}$, are positively identified as ML estimates.

Finally, the proposed method was used to lower the average complexity of the original semidefinite relaxation detector. The average number of floating point operations of the modified SDR (MSDR) was compared to the original SDR as outlined in [6, 7]. In this algorithm problem (5) is solved by an interior point method. In the MSDR this interior point algorithm is applied only in those instances when the ML verification test fails. As above, the ZF estimate was used as an initial estimate of $\mathbf{s}$. In Fig. 1(c) it can clearly be seen that in the regime were the ML verification is useful with high probability, the average complexity of the MSDR detector is significantly lower than that of the original SDR. Note however that the BERs of the SDR and MSDR are identical due to the way in which the MSDR was constructed.

6. CONCLUSION AND DISCUSSION

Herein, an add-on procedure which can be used to lower the average complexity of any ML detection algorithm was proposed and evaluated. It was argued analytically and shown by numerical simulations that the proposed method is able to exploit the property that many instances of the ML detection problem are easy in the high SNR regime and when the channel matrix is well conditioned.

Reducing the average complexity of the ML detector may be useful when several consecutive symbols are detected and when the probability that the algorithm has a complexity which deviates substantially from the average is small. Also, even in the case when the ML detector is implemented under a strict time constraint reducing the average complexity could be a way of reducing hardware power consumption.

It is also worthwhile to mention that the proposed complexity reduction method can be applied to some other related detection problems, such as those in [12, 13].

7. REFERENCES

[1] S. Verdú, "Computational complexity of multiuser detection," *Algorithmica*, vol. 4, pp. 303–312, 1989.

[2] M. Ajtai, "The shortest vector problem in L_2 is NP-hard for randomized reductions," *in Proc. 30th Annu. ACM Symp. Theory Comput.*, pp. 10–19, 1998.

[3] M. O. Damen, H. El Gamal, and G. Caire, "On maximum-likelihood detection and the search for the closest lattice point," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2389–2401, Oct. 2003.

[4] B. Hassibi and H. Vikalo, "Maximum-likelihood decoding and integer least-squares: The expected complexity," in *Multiantenna Channels: Capacity, Coding and Signal Processing*, J. Foschini and S. Verdú, Eds. Amer. Math. Soc., 2003.

[5] J. Jaldén and B. Ottersten, "An exponential lower bound on the expected complexity of sphere decoding," in *Proc. IEEE ICASSP'04*, May 2004.

[6] P.H. Tan and L.K. Rasmussen, "The application of semidefinite programming for detection in CDMA," *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 8, pp. 1442–1449, Aug. 2001.

[7] W.-K. Ma, T. N. Davidson, K.M. Wong, Z.-Q Luo, and P.-C. Ching, "Quasi-maximum-likelihood multiuser detection using semi-definite relaxation with application to synchronous CDMA," *IEEE Transactions on Signal Processing*, vol. 50, no. 4, pp. 912–922, Apr. 2002.

[8] C. Helmberg, F. Rendl, R.J. Vanderbei, and H. Wolkowicz, "An interior-point method for semidefinite programming," *SIAM Journal on Optimization*, vol. 6, pp. 342–361, 1996.

[9] J. Jaldén, C. Martin, and B. Ottersten, "Semidefinite programming for detection in linear systems – optimality conditions and space-time decoding," in *Proc. IEEE ICASSP'03*, Apr. 2003.

[10] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[11] G. W. Stewart, *Matrix Algorithms. Volume 1: Basic Decompositions*, SIAM, 1998.

[12] Q. Li, E.-W. Bai, and Z. Ding, "Blind source separation of signals with known alphabets using ϵ-approximation algorithms," *IEEE Trans. Signal Processing*, vol. 51, no. 1, pp. 1–10, 2003.

[13] W.-K. Ma, P.-C. Ching, T. N. Davidson, and X.-G. Xia, "Blind maximum likelihood decoding of orthogonal space-time block codes: A semidefinite relaxation approach," in *Proc. IEEE 2003 Global Commun. Conf.*, Dec. 2003.