CFAR DETECTION OF A KNOWN FSK-MODULATED SIGNAL IN WHITE GAUSSIAN NOISE WITH UNKNOWN VARIANCE

Ian Atkinson and Farzad Kamalabadi

Department of Electrical and Computer Engineering and Coordinated Science Laboratory
University of Illinois at Urbana-Champaign

ABSTRACT

Optimal detection of a known FSK-modulated binary signal in additive white Gaussian noise using a matched filter receiver requires knowledge of second-order noise statistics. The dependence on noise statistics causes the probability of detection to be sensitive to errors in the noise variance value. This makes optimal scheme of limited use in situations where the noise statistics are unknown and cannot be estimated reliably. We propose an alternative approach, which has a more easily calculated test statistic than the optimal method and yields a constant false detection rate, regardless of noise statistics. While this approach has sub-optimal detection probability, it will be significantly advantageous in applications where a primary interest in to strictly limit false detections. In addition, it allows one to easily determine the required detection length (in symbols) to achieve a desired performance level.

1. INTRODUCTION

Detection of a signal corrupted by additive white Gaussian noise is a classic problem arising in many communication and radar systems. Consider for example a wireless communication device, such as a cellular telephone or personal pager, which upon *power up* must detect and synchronize with a network that indicates its presence by periodically transmitting a known synchronization signal. Unfortunately, in such scenarios the optimal detection method is unable to achieve a constant false detection rate. In this paper, we investigate the detection of a known *binary* frequency shift key (FSK) modulated signal corrupted by additive white Gaussian noise using a matched filter receiver. This known signal could correspond to a network identifier as described above, an asynchronous transmission indicator, or some other signal that must be detected for successful operation.

Optimal detection of a known FSK-modulated signal using a matched filter receiver requires knowledge of the second-order noise statistics and the transmit signal energy in addition to the known signal symbols (number, value, and transmit phase). Although the transmit signal energy will generally be known *a priori* by the receiver, noise statistics (specifically variance) may be unknown and possibly time-varying. This requires the detecting system to estimate this value and calculate the required detection parameters in real-time. Poor estimates of noise variance will cause the actual false alarm rate to differ from the desired value, which may be highly undesirable for certain scenarios.

We propose an alternative detection scheme, which is still based on a matched filter receiver, but requires only the length and values of the known signal to compute the optimal parameters. As a result, a constant false alarm rate (CFAR) is achieved for this approach, allowing the probability of a false detection to be strictly controlled. Achieving this robustness, however, requires sacrificing probability of detection compared to the optimal method. However, a high probability of detection can be achieved by increasing the transmit signal energy or the length of the signal (number of symbols).

Matched filtering of an FSK-modulated signal is reviewed in Section 2 and the statistics of the filtered signals are investigated. Section 3 outlines the optimal approach to this detection problem that uses a linear combination of samples from the filtered signals and requires the noise statistics to optimally make a decision. The proposed method is motivated, derived, and compared to the optimal method in Section 4

2. MATCHED FILTERING OF AN FSK-MODULATED SIGNAL

Detection of an FSK-modulated signal can be formulated as a binary hypothesis testing problem with the hypothesis being the presence and absence of the modulated signal, respectively.

$$H_0: \text{ No signal present (noise only)}$$
$$H_1: \text{ Signal present}$$

Let $b_n \in \{0,1\}$ for $0 \le n < N$ denote the binary (or symbol) representation of the known length-N signal and $\beta_i \in \mathbb{R}$ for $i = 0, 1$ be the frequencies used to represent a zero and one, respectively. We will assume that $\beta_0 \ne \beta_1$ and $(\beta_1 - \beta_0) \in \mathbb{Z}$, although all results presented are approximate when $(\beta_1 - \beta_0) \notin \mathbb{Z}$ but $|\beta_1 - \beta_0| \gg 0$.

The signal at the receiver can be described by

$$s(t) = \sum_{n=0}^{N-1} s\left(t - nT, \beta_{b_n}, \theta_n\right) + W(t) \tag{1}$$

where $W(t)$ is a *complex*[1] white Gaussian noise process with variance $\frac{N_0}{2}$ and $s(\cdot,\cdot,\cdot)$ describes a modulated symbol.

$$s\left(t, \beta_{b_n}, \theta_n\right) = \begin{cases} \sqrt{\frac{2E_s}{T}} e^{j2\pi\beta_{b_n}\frac{t}{T} + j\theta_n} & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_n \in [0, 2\pi)$ is the initial phase of the n^{th} symbol, T is the symbol duration, and E_s is the signal energy. With proper selection of phase values, this model can describe phase-continuous modulation.

[1] $W(t) = W_R(t) + jW_I(t)$ where $W_R(t), W_I(t) \sim N\left(0, \frac{N_0}{2}\right)$ and $\mathrm{E}\left[W_R(t)W_I(s)\right] = 0$ for all t, s.

The matched filters [1] for the signal model described above are given by

$$h_i(t) = s^*(T-t, \beta_i, 0) \qquad i = 0, 1$$

and have the property

$$\int_0^T h_i(t) s(T-t, \beta_{b_k}, \theta_k) dt = 2E_s e^{-j\pi\left(\beta_i - \beta_{b_k}\right)} e^{j\theta_k} \operatorname{sinc}\left(\pi\left(\beta_i - \beta_{b_k}\right)\right), \quad (2)$$

which shows that the b_k^{th} filter has the largest response when the symbol is b_k. Plugging the filter definitions in (2), we see that the matched filters are orthogonal for the assumed conditions $\beta_1 \neq \beta_0$ and $(\beta_1 - \beta_0) \in \mathbb{Z}$ and each have a norm equal to $2E_s$.

Let $Y_i(t)$ to be the output of the i^{th} matched filter at time t.

$$Y_i(t) = \int_0^T h_i(\tau) s(t-\tau) d\tau \qquad i = 0, 1 \quad (3)$$

Notice that for $0 < t \leq NT$, $Y_i(t)$ takes on *meaningful* values at integer multiples of the symbol time, and all other times contain contributions from two symbols. Furthermore, the first symbol, b_0, does not appear at $Y_i(t)$ until $t = T$. We will only consider $Y_i(nT)$ at these times by defining an observation as

$$V(n) = V_R(n) + jV_I(n) = Y_{b_{n-1}}(nT) \quad 1 \leq n \leq N \quad (4)$$

It it straightforward to show that for $1 \leq m, n \leq N$

$$\begin{aligned} H_0: &\ V_R(n) \sim N(0, E_s N_0), V_I(n) \sim N(0, E_s N_0) \\ H_1: &\ V_R(n) \sim N(2E_s \cos\theta_{n-1}, E_s N_0), \\ &\ V_I(n) \sim N(2E_s \sin\theta_{n-1}, E_s N_0) \end{aligned}$$

and $V_R(n)$ is independent of $V_I(m)$.

3. OPTIMAL DETECTION

The detection of a known signal in additive Gaussian noise is a classic and well studied problem [1,2]. As will be seen, the optimal decision rule (when noise variance is known) requires knowledge of the noise variance, which makes the optimal detection method sensitive to changes in second-order noise statistics.

Let X_N denote the length-N vector of observations

$$X_N = \begin{bmatrix} V(1) & V(2) & \cdots & V(N) \end{bmatrix}^T \quad (5)$$

Since $V(n)$ are independent random variables, the likelihood ratio of $X_N = x_N$ is simply the product of the likelihood ratio of $V(n) = v(n)$ for $n = 1, 2, \ldots, N$. The log-likelihood ratio therefore is

$$\begin{aligned} \log \mathrm{L}_{X_N,N}(x_N) &= \log \left(\prod_{n=1}^{N} \frac{p_{V(n)|H_1}(V(n) = v(n)|H_1)}{p_{V(n)|H_0}(V(n) = v(n)|H_0)} \right) \\ &= \frac{2}{N_0} \sum_{n=1}^{N} (v_R(n)\cos\theta_{n-1} + v_I(n)\sin\theta_{n-1} - E_s) \\ &= \frac{2}{N_0}(z_N - NE_s), \end{aligned}$$

where $z_N = \sum_{n=1}^{N}(v_R(n)\cos\theta_{n-1} + v_I(n)\sin\theta_{n-1})$ is a sample value of the random variable

$$Z_N \triangleq \sum_{n=1}^{N}(V_R(n)\cos\theta_{n-1} + V_I(n)\sin\theta_{n-1})$$

The optimal decision rule is of the form

$$Z_N \underset{H_0}{\overset{H_1}{\gtrless}} \tau_N,$$

for some threshold τ_N, that in general will vary with N. To achieve a false alarm probability of α, we employ Neyman-Pearson [2] hypothesis testing by solving

$$\alpha = P(Z_N > \tau_N | H_0) \quad (6)$$

for τ_N. It can be shown that

$$\begin{aligned} H_0: &\ Z_N \sim N(0, NE_s N_0) \\ H_1: &\ Z_N \sim N(2NE_s, NE_s N_0) \end{aligned}$$

which makes the solution to (6)

$$\tau_N = \sqrt{NE_s N_0} Q^{-1}(\alpha) \quad (7)$$

where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ and $Q^{-1}(\cdot)$ is the inverse of this function. The probability of detection for the threshold calculated according to (7) is

$$P_D = \mathrm{P}(Z_N > \tau_N | H_1) = Q \quad Q^{-1}(\alpha) - 2\sqrt{\frac{NE_s}{N_0}}\Bigg) \quad (8)$$

The optimal method for detecting a known FSK-modulated length-N signal is to compare a linear combination of the N observations to a threshold. Selecting the threshold according to (7) allows for a desired probability of false detection to be achieved, given values for N_0 and E_s. The resulting probability of detection can be made arbitrarily large by increasing either the signal energy or the length of the known signal.

4. PROPOSED DETECTION APPROACH

The optimal detection method presented in Section 3 requires one to know or estimate the noise variance in order to calculate the optimal threshold. In addition, in order to calculate the statistic required to compare to this threshold, the transmit phases of the symbols must be known.

In this section, we present our proposed detection scheme that uses a non-linear combination of samples from *both* matched filters to detect the signal. The threshold for this detection scheme has the property that for a desired probability of false detection, the same value is optimal for any noise variance. In addition, the test statistic can be calculated without any phase information, which means that unlike the optimal case, the same receiver could be used to detect phase-continuous and non-phase-continuous modulated signals.

$V(n)$ was defined in (4) as the output of the matched filter containing the signal at $t = nT$. Let $U(n)$ denote output of the *other* matched filter

$$U(n) = U_R(n) + jU_I(n) = Y_{b_{n-1}^C}(nT) \qquad 1 \leq n \leq N$$

where b_n^C is the *complement* of b_n. From the definitions of $V(n)$ and $U(n)$, and the results of Section 2 we can conclude that for either hypothesis, $U(n)$ has the same distribution as $V(n)|H_0$.

Under both hypotheses the real and imaginary parts of $V(n)$ and $U(n)$ are independent Gaussian random variables. Therefore, the magnitude squared of these random variables normalized by their variance are chi-square random variables with two degrees of freedom [3, 4].

$$\begin{aligned} H_0 &: \frac{|U(n)|^2}{E_sN_0} \sim \chi_2^2, \frac{|V(n)|^2}{E_sN_0} \sim \chi_2^2 \\ H_1 &: \frac{|U(n)|^2}{E_sN_0} \sim \chi_2^2, \frac{|V(n)|^2}{E_sN_0} \sim \chi_2^2(\lambda) \end{aligned}$$

where χ_p^2 denotes a chi-square random variable with p degrees of freedom and

$$\lambda = \frac{4E_s^2\left(\cos^2\theta_{n-1} + \sin^2\theta_{n-1}\right)}{E_sN_0} = 4E_s/N_0$$

is the non-centrality parameter that arises only in the alternate hypothesis. By considering $|V(n)|^2$ and $|U(n)|^2$ rather than $V(n)$, the requirements for phase information are removed, which means detection based on these values can be performed without knowledge of θ_n.

Let S_N and W_N denote the sum of the first N magnitudes squared of $V(n)$ and $U(n)$, respectively.

$$S_N = \sum_{n=1}^{N} |V(n)|^2 \qquad W_N = \sum_{n=1}^{N} |U(n)|^2$$

And, since the sum of chi-square random variables is itself a chi-square random variable

$$\begin{aligned} H_0 &: \frac{W_N}{E_sN_0} \sim \chi_{2N}^2, \frac{S_N}{E_sN_0} \sim \chi_{2N}^2 \\ H_1 &: \frac{W_N}{E_sN_0} \sim \chi_{2N}^2, \frac{S_N}{E_sN_0} \sim \chi_{2N}^2(N\lambda) \end{aligned} \tag{9}$$

Recalling that the distribution of test statistic(s) under the null hypothesis is directly related to the probability of false detection, it is apparent from (9) that if E_s and N_0 are available to normalize S_N and W_N, then null hypothesis will depend only on the number of observations. A decision rule for a given false alarm rate could therefore be found that only depends on the number of observations. Removing the dependence of the null hypothesis on E_s and N_0 can be accomplished by noting that the ratio of S_N to W_N, which we denote R_N, can be written as

$$R_N \triangleq \frac{S_N}{W_N} = \frac{\frac{S_N}{2NE_sN_0}}{\frac{W_N}{2NE_sN_0}}$$

We recognize the last term in the equality as the ratio of two chi-square random variates, normalized by their degrees of freedom. Therefore, R_N has an F-distribution [4].

$$\begin{aligned} H_0 &: R_N \sim F_{2N,2N} \\ H_1 &: R_N \sim F_{2N,2N}(N\lambda) \end{aligned} \tag{10}$$

Now we see that $R_N|H_0$ depends only on the length of known signal, N, and not on E_s or N_0. This fact is key in being able to have a single test that is universally optimal for all noise variances. In addition, unlike Z_N, the test statistic R_N can be computed without any phase information as it is composed of magnitudes squared.

Given an observation $R_N = r_N$, the optimal decision rule is of the form

$$\mathrm{L}_{R_N,N}(r_N) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_N \tag{11}$$

where $\mathrm{L}_{R_N,N}(\cdot)$ is the likelihood ratio for hypothesis test in (10). After some algebra, it can be shown that this likelihood ratio has the following expression.

$$\mathrm{L}_{R_N,N}(r) = e^{-\frac{\lambda}{2}} \sum_{k=0}^{\infty} \frac{(2N)_k}{(N)_k} \left(\frac{\lambda r}{2(1+r)}\right)^k \frac{1}{k!} \tag{12}$$

where $(A)_B$ is Pochammer's symbol [3].

Similar to the optimal case, we can achieve a false positive probability of α by solving

$$\alpha = \mathrm{P}\left(\mathrm{L}_{R_N,N}(R_N) \geq \gamma_N | H_0\right) \tag{13}$$

for γ_N. Inspection of (12) and (13) reveals that contrary to what is suggested by (10), calculating the optimal threshold for (11) requires knowledge of N_0 since $\lambda = 4E_s/N_0$. Noting however, that $\mathrm{L}_{R_N,N}(\cdot)$ is a one-to-one, monotonic increasing function, and therefore so is its inverse, an equivalent detection test to (11) is

$$r_N \underset{H_0}{\overset{H_1}{\gtrless}} \bar{\gamma}_N$$

where $\mathrm{L}_{R_N,N}(\bar{\gamma}_N) = \gamma_N$. The optimal threshold value can be found by solving

$$\alpha = \mathrm{P}\left(R_N \geq \bar{\gamma}_N | H_0\right) \tag{14}$$

for $\bar{\gamma}_N$, which we see depends only on the number of signal symbols and not on the signal energy, symbol time, or noise variance. Once the threshold has been found, the corresponding probability of detection is

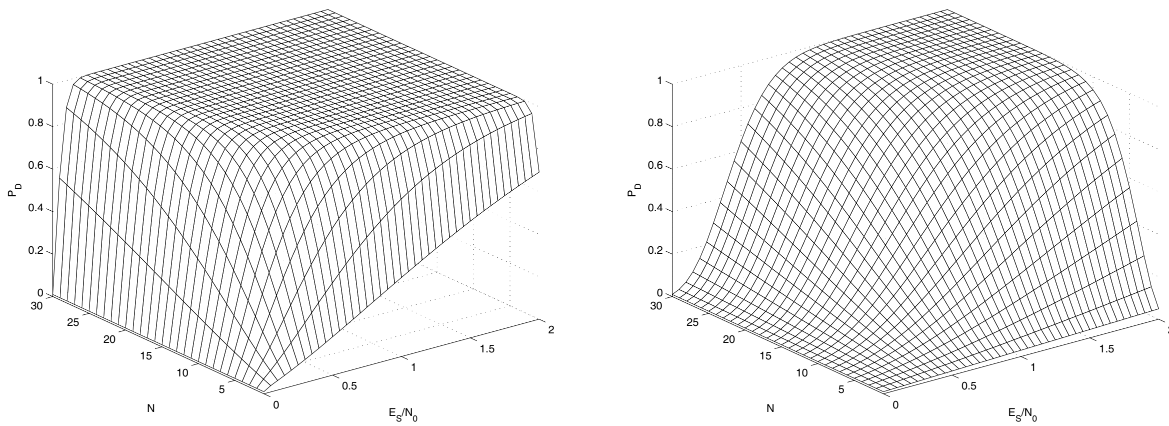
$$P_D = \mathrm{P}\left(R_N \geq \bar{\gamma}_N | H_1\right) \tag{15}$$

Unfortunately, closed form solutions to (14) and (15) do not exist, forcing one to resort to numerical methods or tabulated results [3]. However, provided N and α are fixed, the same threshold is optimal for any E_s and N_0 so the threshold satisfying (14) needs to be found only once. In contrast, the optimal detection method requires a new threshold to be calculated for any change to E_s or N_0.

It is interesting to note that detection performance of the optimal and proposed methods depend on N, E_s and N_0 in much the same way. If we let

$$d^2 = 4N\frac{E_s}{N_0},$$

it is immediately apparent that for α fixed, the probability of detection for the optimal method varies as a function of d while the probability of detection for the proposed method varies as a function of N and d^2. A comparison between the two methods can be made by considering the probability of detection as a two-dimensional function of N and E_s/N_0. Figure 1 shows these two-dimensional *power functions* for the optimal and proposed detection methods, respectively. Although for the same parameters the optimal method provides a higher probability of detection than the proposed method, we can see that by selecting N large enough, the proposed method will yield an acceptable probability of detection.



Optimal Method Proposed Method

Fig. 1. Power functions for the optimal and proposed detection schemes for $\alpha = 0.01$. Graph shows how the probability of detection varies as N and E_s/N_0 change. It can be seen that the optimal detection method provides a high probability of detection except when N is small or $N_0 > 2E_s$. Although the probability of detection for a given N and E_s/N_0 is lower for the proposed method than the optimal method, a high probability of detection can be achieved by selecting N or E_s large enough.

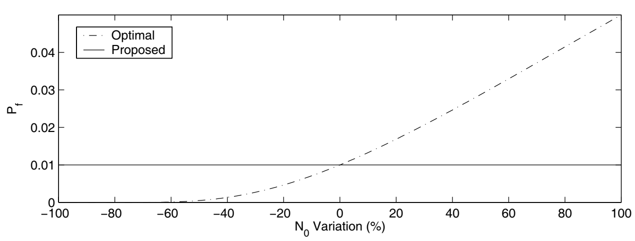


Fig. 2. Sensitivity of the optimal and proposed detection methods to variations in N_0 for $\alpha = 0.01$. As the true N_0 varies from the value used to calculate the thresholds, the *actual* probability of false detection changes greatly for the optimal detection method. The proposed method, however, yields a constant probability of false detection regardless of the actual N_0 value.

The root cause of the optimal method's sensitivity to variations in N_0 is that unlike our proposed method, N_0 appears in the threshold calculations (see (7)). Figure 2 illustrates this sensitivity by showing how the actual probability of false detection for the two methods behaves as the true N_0 changes relative to an estimated N_0 used in calculations. We can see that for the optimal method, an error of 20% in the N_0 parameter results in a change to the probability of false detection of almost 100%. Our proposed detection scheme does not suffer from this issue and achieves exactly the desired false alarm rate for all N_0.

5. CONCLUSIONS

The optimal method of detecting a known FSK-modulated signal in additive white Gaussian noise using a matched filter receiver does not guarantee that a desired probability of false detection will be achieved. This is a direct consequence of the noise variance being used to calculate the threshold for the decision rule in the optimal case. The result is that the false alarm rate is sensitive to changes in the noise variance. Applications which desire to strictly control the probability of false detection are likely to find the optimal method of little use since a change in the noise variance of 20% can translate to a change in the actual false detection probability by almost 100%.

Our proposed method is able to exactly achieve a desired false alarm rate regardless of noise statistics. It accomplishes this feat by considering a sum of values corresponding to the signal divided by a sum of values corresponding to noise only. This ratio is an F-distributed random variable, which, when no signal is present, depends only on the number of symbols in the known signal. The optimal threshold for this scheme is therefore independent of noise variance and transmit signal energy, making the *same* threshold optimal for any values these parameters take. Although the probability of detection that this method yields is sub-optimal, it can be made arbitrarily large by increasing the number of symbols in the known signal being detected. For a fixed performance criteria (probability of detection and false alarm), one can therefore calculate the required length of the detection signal in symbols. The proposed method also removes the need for the transmit phase information to be known *a priori* by the receiver. This allows both phase-continuous and non-phase-continuous FSK-modulated signals to be detected using the same detection scheme and parameters, a feat not possible with the optimal approach.

6. REFERENCES

[1] Harry L. Van Trees, *Detection, Estimation, and Modulation Theory: Part I*, John Wiley and Sons, 1968.

[2] H. Vincent Poor, *An Introduction to Signal Detection and Estimation*, Springer, 2nd edition, 1994.

[3] Milton Abramowitz and Irene A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, 1964.

[4] Alan Stuart, Keith Ord, and Steven Arnold, *Kendall's Advanced Theory of Statistics, Vol 2A: Classical Inference and the Linear Model*, Arnold, 6th edition, 1999.