OPTIMAL DESIGN OF SPECTRUM CONSTRAINED SIGNAL SETS WITH CORRELATION ANALYSIS

*Tao Wang, Weiyong Yan**

Dept of Electrical and Computer Engineering
Curtin University of Technology
GPO Box U 1987, WA 6840, Australia

Zhuquan Zang†

WATRI
35 Stirling Hwy, Crawley WA 6009
Australia

ABSTRACT

This paper is concerned with the design of an optimal set of analog signals with prescribed magnitude spectrum and quadratic phase structure such that the maximum cross-correlation is minimized. An analytic expression for the maximum cross-correlation between two signals is derived through mathematical analysis. The optimal set of signals with the lowest maximum cross-correlation is explicitly characterized under certain conditions.

1. INTRODUCTION

The design of signal sets with prescribed spectral properties and low values of correlation is an important element of modern multidimensional signaling and multiuser communications system design [3, 6, 9]. Such a design also plays a vital role in many other areas of signal processing such as multi-target detection in radar and sonar systems [1,3,7]. Considerable effort has been devoted to synthesizing signal sets with low values of cross-correlation at all lags and low values of auto-correlation at nonzero lags [2, 8]. Successful design of signal sets with these characteristics is desirable because of the needs for increasing the number of simultaneous access users and the reduction of inter-symbol interference and co-channel interference in Code Division Multiple Access (CDMA) communication systems [5, 9]. For target detection in radar and sonar systems, the signal-to-noise ratio (SNR) can be improved significantly by the proper choice of signal sets [1]. In order to maximize the SNR at the output of the receiver, one effective approach is to shape the transmitted signal to be the inverse backscattering spectrum [1], which has an approximately rectangular shape in both time and frequency domains. In [3], a specific set of signals, which have unit energy, a constant passband magnitude and quadratic phase structure in frequency domain, has been investigated. The reason for designing such signals is that the shapes of the complex envelopes of the signals are approximately rectangular in both time and frequency domain, and the magnitude of the cross-correlation function of the signal set can be less than that of other signal sets such as Gold sequences or Kasami sequences [8]. The signal sets characterized in [3] have the property that the coefficients of their quadratic phase function lie on an ellipse which is uniquely determined by the bandwidth and time duration of the signals. As such, for a given signal set with specific bandwidth and time duration, the cross-correlation magnitude becomes a major criterion for measuring the performance and quality of the signal set. In practice, it is always desirable to make the maximum cross-correlation of a signal set as small as possible. In [3], a very simple upper bound for the maximum cross-correlation between two signals was established, although no method for designing an optimal signal set was given. In [4], it was further shown that the upper bound can be minimized simply by selecting the coefficients in such a way that they are equally spaced along the horizontal axis of an ellipse. In addition, a new and tighter upper bound on the maximum cross-correlation magnitude between two signals was derived. However, this bound is almost as hard to compute as the original maximum cross-correlation.

In this paper, the problem of computing and minimizing the maximum cross-correlation magnitude is investigated. It is shown that under certain conditions on the design parameters, the maximum cross-correlation magnitude itself can be expressed in closed form. In other words, an analytic expression for the maximum cross-correlation can be mathematically established which allows an efficient design method to be devised for designing a set of optimal signals with the prescribed spectrum and correlation properties.

2. PRELIMINARIES

2.1. Signal set design problem

The signal set design problem is to find a set of signals which possess prescribed properties in both time and frequency domains. Mathematically, the signal set design problem [3] can be stated as follows. Design a set of signals $\mathbf{s} = \{s_i(t);\ i = 1, 2, \cdots, N\}$ defined over $\left[-\frac{T}{2},\ \frac{T}{2}\right]$ with their corresponding Fourier transforms $\mathbf{S} = \{S_i(f);\ i = 1, 2, \cdots, N\}$ satisfying the following properties:

$$\int_{-\frac{T}{2}}^{+\frac{T}{2}} |s_i(t)|^2\, dt = 1; \qquad i = 1, 2, \cdots, N \tag{1}$$

$$|R_{i,j}(\tau)| \le \delta; \qquad -T \le \tau \le T;\ \ i \ne j \tag{2}$$

$$|S_i(f)| = \begin{cases} \alpha(f); & |f| \le W; \\ \varepsilon_i(f); & |f| > W; \end{cases} \qquad i = 1, 2, \cdots, N \tag{3}$$

where $R_{i,j}(\tau)$ is the cross-correlation function between signals $s_i(t)$ and $s_j(t)$ defined by

$$R_{i,j}(\tau) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} s_i(t) s_j^*(t - \tau) dt; \qquad -T \le \tau \le T \tag{4}$$

*The work of this author was partially supported by a research grant from the Australian Research Council

†The work of this author was partially supported by a research grant from Curtin University of Technology

δ is a constant which depends on the time-bandwidth product and the number of signals in the set, $\alpha(f)$ is a constant function and $\varepsilon_i(f)$ is a signal with approximately zero energy. It has been argued in [3, 7] that signals with a constant magnitude spectrum in the passband and a quadratic phase structure are good candidates from both practical and theoretic points of view. Specifically, in frequency domain such signals can be expressed in the form

$$S_i(f) = \begin{cases} |S_i(f)|\, e^{j\left(a_i f^2 + b_i f + c_i\right)}; & |f| \le W \\ \varepsilon_i(f); & |f| > W \end{cases} \tag{5}$$

where

$$|S_i(f)|_{|f|\le W} = \sqrt{\frac{1-\varepsilon}{2W}} \approx \frac{1}{\sqrt{2W}}$$

and the phase coefficients a_i and b_i are located on the ellipse described below,

$$\frac{a_i^2}{\left(\frac{\pi T}{2W}\right)^2} + \frac{b_i^2}{\frac{(\pi T)^2}{3}} = 1 \tag{6}$$

where T is a given time duration and $2W$ is a given bandwidth. In this family of signals, every signal satisfies (1), (3) and is parameterized by three coefficients a_i, b_i, c_i with a_i, b_i on the ellipse (6). Moreover, a simple upper bound is derived in [3] on the maximum magnitude of the cross-correlation between any two signals in the class. Based on the upper bound a suboptimal algorithm is proposed in [4] which can be used effectively to design a set of signals with the prescribed properties.

2.2. Estimate of maximum cross-correlation magnitude

It is shown in [3, 4] that the cross-correlation function (4) can be approximately expressed as

$$R_{i,j}(\tau) = \frac{1}{2W} e^{j\Delta c} \int_{-W}^{+W} e^{j\left[\Delta a f^2 + (\Delta b + 2\pi\tau) f\right]} df \tag{7}$$

where $\Delta a = a_i - a_j > 0; \Delta b = b_i - b_j; \Delta c = c_i - c_j$. Furthermore, the maximum cross-correlation magnitude can be represented by a pair of Fresnel integrals as follows

$$\begin{aligned} &\max_{-T\le\tau\le T} |R_{i,j}(\tau)| \\ &= \sqrt{\frac{\pi}{8\Delta a W^2}} \max_{-T\le\tau\le T} \left\{ \left[S(x(\tau)+y) - S(x(\tau)-y)\right]^2 \right. \\ &\quad \left. + \left[C(x(\tau)+y) - C(x(\tau)-y)\right]^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{8}$$

where

$$y = W\sqrt{\frac{2\Delta a}{\pi}}, \quad x(\tau) = \sqrt{\frac{2}{\pi \Delta a}} \frac{\Delta b + 2\pi\tau}{2} \tag{9}$$

and the Fresnel integral functions $C(x)$ and $S(x)$ are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \tag{10}$$

From the definition of $x(\tau)$ in (9), we have, for $-T \le \tau \le T$

$$\sqrt{\frac{2}{\pi\Delta a}} \frac{\Delta b - 2\pi T}{2} \le x(\tau) \le \sqrt{\frac{2}{\pi\Delta a}} \frac{\Delta b + 2\pi T}{2};$$

Let

$$X_0 = \sqrt{\frac{2}{\pi\Delta a}} \frac{\Delta b - 2\pi T}{2}, \quad X_1 = \sqrt{\frac{2}{\pi\Delta a}} \frac{\Delta b + 2\pi T}{2} \tag{11}$$

It follows from (8) that

$$\begin{aligned} &\max_{-T\le\tau\le T} |R_{i,j}(\tau)| \\ &= \sqrt{\frac{\pi}{8\Delta a W^2}} \max_{X_0\le x\le X_1} \left\{ \left[S(x+y) - S(x-y)\right]^2 \right. \\ &\quad \left. + \left[C(x+y) - C(x-y)\right]^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{12}$$

As the Fresnel integral pair $C(x)$ and $S(x)$ are continuous and differentiable, a solution to the following maximization problem exists over the interval $[X_0, X_1]$.

$$\max_x \left\{ \left[S(x+y) - S(x-y)\right]^2 + \left[C(x+y) - C(x-y)\right]^2 \right\}$$

In the next section, we will concentrate on the study of the analytic properties of the maximum cross-correlation magnitude function.

3. ANALYTIC FORMULA FOR MAXIMUM CROSS-CORRELATION

An approach to estimating the maximum cross-correlation magnitude was proposed in [4], where some approximations were introduced to facilitate the estimation procedures but led to a suboptimal design. In this section, some important technical lemmas will be summarized. These results then will be applied to find an analytic expression for the maximum cross-correlation magnitude function. Define

$$f(x,y) \triangleq \left[S(x+y) - S(x-y)\right]^2 + \left[C(x+y) - C(x-y)\right]^2 \tag{13}$$

It can be seen that

$$\max_{-T\le\tau\le T} |R_{i,j}(\tau)|^2 = \frac{\pi}{8\Delta a W^2} \max_{X_0\le x\le X_1} f(x,y) \tag{14}$$

where y is defined in (9). Clearly, for any given y the maximum cross-correlation magnitude (as a function of x) is determined by the maximum values of the function $f(x,y)$ under certain conditions. Therefore, it will be beneficial to analyze and compute the maxima and minima of the function $f(x,y)$ with respect to x (for any fixed y). Let us introduce

$$g_y(x) \triangleq f(x,y) \tag{15}$$

for any fixed y. The properties of $g_y(x)$ are summarized through a series of technical lemmas as follows (see [10] for details).

Lemma 1 *$g_y(x)$ is an even function, that is, $g_y(-x) = g_y(x)$.*

Lemma 2 *For any given y, x is an extreme point of the function $g_y(x)$ if and only if x is expressible as $x = \frac{k}{y}$ for some integer k or satisfies the following equation*

$$\int_0^y \cos(\pi x\theta) \sin\left[\frac{\pi}{2}(\theta^2 - y^2)\right] d\theta = 0$$

Lemma 3 *Under the condition $0 < y \le 1$, $x = 0$ is the only extreme point of the function $g_y(x)$ in the interval $\left(-\frac{1}{y}, \frac{1}{y}\right)$, that is*

$$g_y(0) = \max_{-\frac{1}{y}<x<\frac{1}{y}} g_y(x)$$

Lemma 4 *Under the condition* $1 < y \leq \sqrt{2}$, $x = 0$ *is the unique maximum point of the function* $g_y(x)$ *in the interval* $\left(-\frac{1}{y}, \frac{1}{y}\right)$, *that is*

$$g_y(0) = \max_{-\frac{1}{y}<x<\frac{1}{y}} g_y(x)$$

We believe that the above lemmas are of interested in their own right in the study of Fresnel functions which have applications in many signal processing areas. Lemma 1 enables us to concentrate on the investigation of the function $g_y(x)$ for $x \geq 0$. Lemma 2 describes the location of extreme points of the function $g_y(x)$. Lemmas 3 and 4 characterize the extreme points of $g_y(x)$ around the origin subject to a constraint on y.

Using Lemmas 1 to 4, the following main theorem (see [10] for details) can be established. It demonstrates that the maximum cross-correlation magnitude can be accurately determined and represented in an analytic expression when some conditions imposed on y are satisfied. To simplify notation, let us define

$$R(\tau) \triangleq R_{i,j}(\tau) \tag{16}$$

where $R_{i,j}$ denotes the cross-correlation function between arbitrary two signals as defined in (7).

Theorem 1 *For any given* Δa *and* Δb *with*

$$|\Delta a| \leq \frac{\pi}{W^2} \quad \textit{and} \quad |\Delta b| \leq \left|\frac{\pi}{W} - 2\pi T\right| \tag{17}$$

there holds

$$\max_{-T\leq\tau\leq T} |R(\tau)| = \left|R\left(-\frac{\Delta b}{2\pi}\right)\right|$$

$$= \frac{1}{W\sqrt{\frac{2|\Delta a|}{\pi}}}\left[S^2\left(W\sqrt{\frac{2|\Delta a|}{\pi}}\right) + C^2\left(W\sqrt{\frac{2|\Delta a|}{\pi}}\right)\right]^{\frac{1}{2}} \tag{18}$$

where $S(\cdot)$, $C(\cdot)$ *are Fresnel functions as defined in (10).*

4. DESIGN OF AN OPTIMAL SIGNAL SET

It has been shown in the previous section that the maximum of the cross-correlation magnitude function achieves its maximum at the origin and the maximum can be accurately represented as a function of y over the given interval $\left[-\frac{1}{y}, \frac{1}{y}\right]$ under certain conditions. In this section, it will be shown that in order to minimize the maximum cross-correlation magnitude function between any two signals in the set, the coefficients a_i in the phase functions should be equally spaced along one axis of the ellipse defined by (6).

To illustrate some typical features of the maximum cross correlation magnitude $|R(\tau)|$, Fig. 1 shows the various locations of the maximum points of the cross-correlation magnitudes with respect to different values of y. It can be seen that when $y = 0.5$ and $y = 2$, the maximum cross-correlation magnitude achieve its maximum at $x = 0$. On the other hand, from Figs. 2 we see that the maxima will move further away from the origin with the increase of y. These graphic illustrations are consistent with the theoretical analysis of the cross-correlation magnitude. Therefore it is reasonable to impose certain constraints on y in the design of the signal set in order to effectively minimize the maximum cross-correlation magnitude. Theoretical analysis can be carried out to show that even if in the situations as shown in Fig. 2 (multi-maximum cases), the difference between $g_y(0)$ and $\max|R(\tau)|$ will converge to a fixed constant as y increases (see [10]).

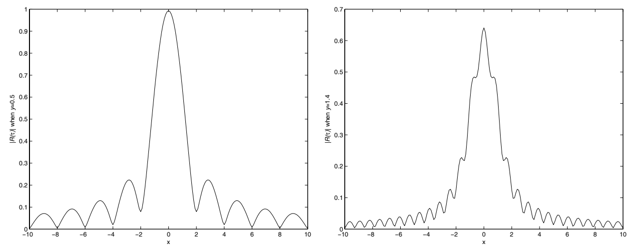

Fig. 1. Plot of $R(\tau)$ when $y = 0.5$ and $y = 1.2$.

Definition 1 *Define*

$$\Re\{S_1, \cdots, S_N\}$$

as the maximum cross-correlation between arbitrary two signals $S_i(f)$ *and* $S_j(f)$ *in a signal set* $\mathbf{S}_N = \{S_1, \cdots, S_N\}$ *for all* τ, *where the signals are expressed in frequency domain, that is*

$$\Re\{S_1, \cdots, S_N\} = \max_{\substack{1\leq i,j\leq N\\ i\neq j}} \max_{\tau} |R_{i,j}(\tau)| \tag{19}$$

The signal set design problem is to find N signals, such that the maximum cross-correlation magnitude amongst the N signals is minimized subject to the constraints that the phase coefficient a_i and b_i satisfy the ellipse equation (6) in the signal space. To clearly demonstrate the signal set design problem, let us consider the following

Problem A: Find a signal set $\bar{\mathbf{S}}_N = \left\{\bar{S}_1, \cdots, \bar{S}_N\right\}$ which solves the following minimization problem

$$\min_{\{S_1,\cdots,S_N\}} \Re\{S_1, \cdots, S_N\} \tag{20}$$

subject to the specified constraints.

We have shown that under certain conditions, the maximum cross-correlation magnitude can be expressed as

$$\Re\{S_1, \cdots, S_N\} = \left\{\max_{\substack{1\leq i,j\leq N\\ i\neq j}} \frac{S^2(y) + C^2(y)}{y^2}\right\}^{\frac{1}{2}} \tag{21}$$

where $y = W\sqrt{\frac{2\Delta a}{\pi}}$ and $\Delta a = |a_i - a_j|$. Introducing

$$\alpha_i = \frac{2W^2}{\pi}a_i; \quad \alpha_j = \frac{2W^2}{\pi}a_j; \quad \text{and} \quad \Delta\alpha = \alpha_i - \alpha_j \tag{22}$$

we have

$$\Re\{S_1, \cdots, S_N\} = \tag{23}$$

$$\left\{\max_{\substack{1\leq i,j\leq N\\ i\neq j}} \frac{S^2(\sqrt{|\alpha_i - \alpha_j|}) + C^2(\sqrt{|\alpha_i - \alpha_j|})}{|\alpha_i - \alpha_j|}\right\}^{\frac{1}{2}} \tag{24}$$

To simplify presentation, define

$$Q(y) \triangleq \frac{S^2(\sqrt{|y|}) + C^2(\sqrt{|y|})}{|y|^2} \tag{25}$$

Let

$$\wp(\alpha_1, \cdots, \alpha_N)$$

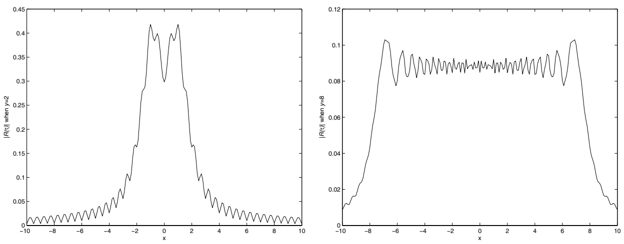

Fig. 2. Plot of $R(\tau)$ when $y = 2$ and $y = 8$. The extrema move further away from the origin.

denote the maximum value of the function $Q(\alpha_i - \alpha_j)$, where $1 \leq i, j \leq N$. that is,

$$\wp(\alpha_1, \cdots, \alpha_N) = \max_{\substack{1 \leq i,j \leq N \\ i \neq j}} Q(\alpha_i - \alpha_j) \qquad (26)$$

Problem A can be can be restated as the following parameterized signal set design problem.

Problem B: Find a set of $\bar{\alpha}_1, \cdots, \bar{\alpha}_N$, such that $\wp(\bar{\alpha}_1, \cdots, \bar{\alpha}_N)$ is minimized subject to the specified constraints, this is,

$$\min_{-WT \leq \alpha_i \leq WT} \wp(\alpha_1, \cdots, \alpha_N)$$

The following theorems and lemma can be established (see [10] for details).

Theorem 2

1. *The function $Q(y)$ as defined in (25) is differentiable everywhere;*
2. *If $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_N\}$ is an optimal solution to Problem B, then an optimal solution to Problem A is given by*

$$\bar{\mathbf{S}}_N = \{\bar{S}_1(f), \cdots, \bar{S}_N(f)\}$$

where the phase coefficients $\bar{a}_i$ are determined by $\bar{\alpha}_i$

From Theorem 2 we see that Problem A is equivalent to Problem B. As a result, the solution to Problem B can be uniquely mapped to the solution to Problem A. Specifically, the solution $\bar{\alpha}_i$ to Problem B determines the solution $\bar{a}_i$ to Problem A by (22).

Lemma 5 *If $0 < y \leq \sqrt{2}$, then, the function $Q(y)$ is monotonically decreasing, that is, $Q'(y)|_{0<y\leq\sqrt{2}} < 0$.*

Theorem 3 *(see [10]) Let a_i and b_i satisfy the elliptic equation (6). If there hold*

$$|b_i - b_j| \leq \left|\frac{\pi}{W} - 2\pi T\right| \quad \text{and} \quad |a_i - a_j| \leq \frac{\pi}{W^2} \qquad (27)$$

then, the maximum cross-correlation magnitude in the signal set can be minimized by choosing the phase coefficients a_i to be equally spaced over the interval $\left[-\frac{\pi T}{2W}, \frac{\pi T}{2W}\right]$.

For the given signal set design problem, Theorem 3 shows that the maximum cross-correlation can be minimized by the proper choice of one of the phase coefficients a_i on the ellipse (6). Particularly, a_i should be equally spaced along the axis within the ellipse subject to the constraint that both a_i and b_i satisfy the elliptic equation.

5. CONCLUDING REMARKS

Many signal processing and multiuser communications applications require to design a set of signals with prescribed spectral properties and low values of cross-correlation magnitude between signals in the set. This paper theoretically analyzes the characteristics of the cross-correlation magnitude where the signals in the set are characterized a constant magnitude and a quadratic phase. It is mathematically proved that the maximum cross-correlation magnitude can be accurately determined under certain conditions. A new method is outlined which can be used for optimal design of a set of signals with the prescribed time and frequency domain properties. In particular, we have shown that to minimize the maximum cross-correlation magnitude under certain conditions, one of the coefficients of the phase function should be equally spaced along the horizontal axis within an ellipse. More work need to be done to extend the result to more common situations by relaxing the bound constraints imposed in the process of establishing the theoretical results in the paper.

6. REFERENCES

[1] M. R. Bell, "Information theory and radar waveform designs," *IEEE Trans. Inform. Theory*, vol. 39, no. 5, pp. 1578-1597, Sep. 1993.

[2] N. B. Chakrabarti and M. Tomlinson, "Design of sequences with specified autocorrelation and cross correlation," *IEEE Trans. Commun*., vol. 24, no. 11, pp. 1246-1252, Nov. 1976.

[3] G. Chandran and J. S. Jaffe, "Signal set design with constrained amplitude spectrum and specified time-bandwidth product," *IEEE Trans. Commun*., vol. 44, no. 6, pp.725-732, Jun. 1996.

[4] B. Jiao, S. Nordholm, W. Y. Yan, and Z. Zang, "On the design of signal sets with constrained magnitude spectrum and phase struncture," *International Journal of Adaptive Control and Signal Processing*, vol. 16, pp. 603-617, 2002.

[5] A. Klapper and C. Cartel, "Spectral methods for cross correlation of geometric sequences," *IEEE Trans. Inform. Theory*, vol. 50, no. 1, pp. 229-232, Jan. 2004.

[6] E.A. Lee and D.G. Messerschmitt, *Digital Communication* (2nd Ed), Boston: Kluwer Academic Publishers, 1994.

[7] A.W. Rihaczek, *Principles of High-Resolution Radar,* Monterey, CA: Peninsula, 1985.

[8] D. V. Sarwate and M. B. Pursley, "Cross-correlation properties of pseudorandom and related sequences," *Proc. IEEE*, vol. 68, no. 5, pp. 593-619, 1980.

[9] S. Verdu, *Multiuser Detection,* Cambridge University Press, 1998.

[10] T. Wang, Y.Y. Yan, and Z. Zang, "Optimal design of spectrum constrained signal sets with correlation analysis," to be submitted.