PREDICTION OF FADING ENVELOPES WITH DIFFUSE SPECTRA

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ABSTRACT

We show how to compute the minimum mean squared prediction error of a mobile-radio fading envelope modeled as a stationary process arising from a diffuse set of local scatterers. Our approach reduces the prediction problem to an eigenvalue decomposition, and is appropriate when the prediction is to be based on error-corrupted estimates of a narrowband fading process. Such prediction can improve the performance of adaptive modulation techniques that require up-to-date channel-state information for optimal performance.

1. INTRODUCTION

In mobile radio, adaptive modulation requires the use of fading-envelope prediction to overcome delays in feeding channel-state information back from receiver to transmitter [1, 2]. In this paper, we show how to compute the minimum mean squared prediction error of a fading envelope modeled as a stationary process arising from a diffuse set of local scatterers. The prediction is based on estimates of the fading envelope over a finite interval of past values, with the estimation errors on this interval modeled as white noise. Our approach reduces the prediction problem to an eigenvalue decomposition [3, p. 242]. This has advantages over the more-common spectrum-factorization approach [3, Sec. 11-6] when the prediction is to be based on error-corrupted estimates of a narrowband fading process, because the spectrum of such a process is nearly discontinuous, and is thus difficult to approximate with a rational function. Example curves are shown for a fading envelope having a Clarke-type Doppler spectrum [4].

2. PREDICTION OF THE FADING ENVELOPE

Let $x(t)$ be a quadrature component of a fading envelope, modeled as a zero-mean, wide-sense stationary random process, and let

$$\tilde{x}(t) = x(t) + v(t), \tag{1}$$

where $\tilde{x}(t)$ is an estimate of past values of the fading envelope and $v(t)$ represents the estimation errors. These errors are modeled as zero-mean white Gaussian noise with variance σ_v^2, and are uncorrelated with $x(t)$. We want to predict the value of $x(r)$ for arbitrary r, given a known, T-length segment of $\tilde{x}(t)$, τ seconds in the past, using a linear predictor

$$\hat{x}(r) = \int_{r-\tau-T}^{r-\tau} \tilde{x}(t)h(r-t)dt. \tag{2}$$

We seek a function $h(t)$ that minimizes the mean squared prediction error,

$$J = \mathcal{E}\{[x(r) - \hat{x}(r)]^2\}. \tag{3}$$

Because of the stationarity of $\tilde{x}(t)$, J does not depend on r. Thus, making use of (1), (2) and (3), and letting $r = \tau + \frac{T}{2}$ we have

$$\begin{aligned} J = \; & \mathcal{E}\left\{\left[x(\tau+\tfrac{T}{2}) - \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)h(\tau+\tfrac{T}{2}-t)dt\right]^2\right\} \\ = \; & \mathcal{E}\left\{\left[x(\tau+\tfrac{T}{2}) - \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)h(\tau+\tfrac{T}{2}-t)dt\right.\right. \\ & \left.\left. - \int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)h(\tau+\tfrac{T}{2}-t)dt\right]^2\right\} \\ = \; & \mathcal{E}\left\{\left[x(\tau+\tfrac{T}{2}) - \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)h(\tau+\tfrac{T}{2}-t)dt\right]^2\right\} \\ & + \mathcal{E}\left\{\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)h(\tau+\tfrac{T}{2}-t)dt\right]^2\right\} \end{aligned} \tag{4}$$

because $x(t)$ and $v(t)$ are uncorrelated. Note that we have chosen r such that the limits of integration in (4) are symmetric. This is advantageous since it leads to a symmetric eigenvalue problem in what follows.

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3. PROBLEM TRANSFORMATION

We start by expressing $h(t)$ as an expansion,

$$h(t) = \sum_{n=0}^{\infty} a_n \phi_n(\tau + \tfrac{T}{2} - t) \quad t \in [\tau, \tau + T]. \tag{5}$$

with $\{\phi_n(t)\}$ the countable orthogonal solutions of

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi(s) R_{xx}(t-s) ds = \lambda \phi(t), \tag{6}$$

where $R_{xx}(t)$ is the autocorrelation of $x(t)$. In keeping with the convention of Slepian, et al. [5], we scale each $\phi_n(t)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n^2(t) dt = \lambda_n. \tag{7}$$

The expansion (5) is valid if $\{\phi_n(t)\}$ is complete on $t \in [-\frac{T}{2}, \frac{T}{2}]$. This is assured if $R_{xx}(t)$ is positive definite [3, p. 374]; that is, if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(t-s) f(s) f^*(t)\, ds\, dt > 0 \tag{8}$$

for any $f(t)$ with

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt > 0. \tag{9}$$

To determine a sufficient condition to satisfy (8), let $f(t) = 0$ for $t \notin [-\frac{T}{2}, \frac{T}{2}]$. Then, writing $R_{xx}(t)$ as the inverse Fourier transform of the power spectral density, $S_{xx}(\omega)$, and changing the order of integration, it is not difficult to show that

$$\begin{aligned} &\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(t-s) f(s) f^*(t)\, ds\, dt \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) |F(\omega)|^2 d\omega, \end{aligned} \tag{10}$$

where $F(\omega)$ is the Fourier transform of $f(t)$. Now $f(t)$ it time limited and, from (9), not identically zero. Thus we can have $F(\omega) = 0$ only on a set of zero measure. Hence, if $S_{xx}(\omega) > 0$ on a set of positive measure, (10) shows that (8) is satisfied. This condition on $S_{xx}(\omega)$ is met whenever the fading envelope is modeled as arising from a diffuse set of local scatterers, as it often is when the number of such scatterers is considered to be large, and thus not individually resolvable (see e.g. Clarke [4]).

Bearing this restriction in mind, and letting P_{xx} represent the power of $x(t)$, we have

$$\begin{aligned}
&\mathcal{E}\left\{ \left[x(\tau + \tfrac{T}{2}) - \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) h(\tau + \tfrac{T}{2} - t) dt \right]^2 \right\} \\
&= \mathcal{E}\Big\{ x^2(\tau + \tfrac{T}{2}) \\
&\qquad - 2\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau + \tfrac{T}{2}) x(t) h(\tau + \tfrac{T}{2} - t)\, dt \\
&\qquad + \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x(s) h(\tau + \tfrac{T}{2} - t) \\
&\qquad\qquad \cdot h(\tau + \tfrac{T}{2} - s)\, dt\, ds \Big\} \\
&= P_{xx} - 2\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(\tau + \tfrac{T}{2} - t) h(\tau + \tfrac{T}{2} - t)\, dt \\
&\quad + \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(t-s) h(\tau + \tfrac{T}{2} - t) \\
&\qquad \cdot h(\tau + \tfrac{T}{2} - s)\, dt\, ds \\
&= P_{xx} \\
&\quad - 2\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(\tau + \tfrac{T}{2} - t) \sum_{n=0}^{\infty} a_n \phi_n(t)\, dt \\
&\quad + \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(t-s) \\
&\qquad \cdot \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n a_p \phi_n(t) \phi_p(s)\, dt\, ds \\
&= P_{xx} \\
&\quad - 2\sum_{n=0}^{\infty} a_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n(t) R_{xx}(\tau + \tfrac{T}{2} - t)\, dt \\
&\quad + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n a_p \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n(t) \\
&\qquad \cdot \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_p(s) R_{xx}(t-s)\, ds \right] dt \\
&= P_{xx} - 2\sum_{n=0}^{\infty} a_n \lambda_n \phi_n(\tau + \tfrac{T}{2}) \\
&\quad + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n a_p \lambda_p \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n(t) \phi_p(t)\, dt \\
&= P_{xx} - 2\sum_{n=0}^{\infty} a_n \lambda_n \phi_n(\tau + \tfrac{T}{2}) + a_n^2 \lambda_n^2,
\end{aligned} \tag{11}$$

where (5), (7) and the orthogonality of $\{\phi_n(t)\}$ were used, and where each $\phi_n(t)$ is defined for all t by (6).

Considering now the noise term of (4) we have

$$\begin{aligned}
\mathcal{E}&\left\{\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)h(\tau+\tfrac{T}{2}-t)dt\right]^2\right\} \\
&= \sigma_v^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} h^2(\tau+\tfrac{T}{2}-t)dt \\
&= \sigma_v^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\sum_{n=0}^{\infty} a_n\phi_n(t)\right]^2 dt \\
&= \sigma_v^2 \sum_{n=0}^{\infty} a_n^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n^2(t)dt \\
&= \sigma_v^2 \sum_{n=0}^{\infty} a_n^2\lambda_n. \qquad (12)
\end{aligned}$$

4. SOLUTION OF THE PREDICTOR

Combining (4), (11), and (12) we have

$$J = P_{xx} + \sum_{n=0}^{\infty} -2a_n\lambda_n\phi_n(\tau+\tfrac{T}{2}) + a_n^2\lambda_n(\lambda_n+\sigma_v^2). \quad (13)$$

Setting the partial derivatives to zero we have

$$\frac{\partial J}{\partial a_k} = -2\lambda_k\phi_k(\tau+\tfrac{T}{2}) + 2a_k\lambda_k(\lambda_k+\sigma_v^2) = 0. \quad (14)$$

Solving for a_k yields

$$a_k = \frac{\phi_k(\tau+\frac{T}{2})}{\lambda_k+\sigma_v^2}. \quad (15)$$

This may be substituted into (5) to obtain $h(t)$. Thus, the predictor (2) is determined.

Note that the expression (15) for the coefficients $\{a_k\}$ has the same form as the expression we previously found for bandlimited processes with flat spectral densities [6, eq. 14]. In that case, σ_v^2 was interpreted as a Lagrange multiplier arising out of an energy constraint. The basis functions $\{\phi_n(t)\}$ were the prolate spheroidal wave functions [5], which are the solutions of (6) when

$$R_{xx}(t) = \frac{\sin\Omega t}{\pi t}, \quad (16)$$

with Ω the band limits of the process $x(t)$. The solution presented in this paper generalizes those results, by accommodating processes with a wider class of spectral densities, including non-bandlimited processes. On the other hand, since mobile-radio fading envelopes are often modeled as bandlimited [4, 7, 8], it is important to note that for such processes, the modeling of estimation errors $v(t)$ in (4) is critical, since the optimization problem otherwise fails to have a solution [9].

When noise is modeled, the predictor problem can be solved by factoring the spectrum [3, Sec. 11.6], as is often done in Wiener problems. But for bandlimited and narrowband fading envelopes, this approach is difficult, because the signal-plus-noise spectrum is nearly discontinuous, and is not easy to approximate using a rational function.

We may also substitute (15) into (13) to get the minimum mean square prediction error,

$$J_{\min} = P_{xx} - \sum_{n=0}^{\infty} \lambda_n \frac{\phi_n^2(\tau+\frac{T}{2})}{\lambda_n+\sigma_v^2}. \quad (17)$$

5. COMPUTING BASIS FUNCTION VALUES

The basis functions $\{\phi_n(t)\}$ and eigenvalues $\{\lambda_n\}$ are solutions of the integral eigenvalue problem (6). We compute these solutions using the Nystrom method [10, p. 791]. First, we approximate the integral in (6) with a quadrature rule

$$\sum_{j=1}^{N} w_j\phi(t_j)R(t-t_j) = \lambda\phi(t), \quad (18)$$

where $\{w_j\}$ are the weights of the quadrature rule and $\{t_j\}$ are the quadrature points. Now we evaluate this equation at the quadrature points,

$$\sum_{j=1}^{N} w_j\phi(t_j)R(t_i-t_j) = \lambda\phi(t_i) \quad i = 1, 2, \ldots, N. \quad (19)$$

Next, we let K be an $N \times N$ matrix with

$$[K]_{i,j} = w_jR(t_i-t_j), \quad (20)$$

and we let ϕ be an $N \times 1$ vector with

$$\phi = [\phi(t_1)\,\phi(t_2)\,\ldots\,\phi(t_N)]^T. \quad (21)$$

Then (19) may be written as a matrix eigenvalue problem

$$K\phi = \lambda\phi. \quad (22)$$

This problem has N eigenvectors $\{\phi_n\}$ and N eigenvalues $\{\lambda_n\}$, which may be determined numerically and substituted into (18). The resulting equation may be solved for the eigenfunction

$$\phi_n(t) = \frac{1}{\lambda_n}\sum_{j=1}^{N} w_j\phi_n(t_j)R(t-t_j). \quad (23)$$

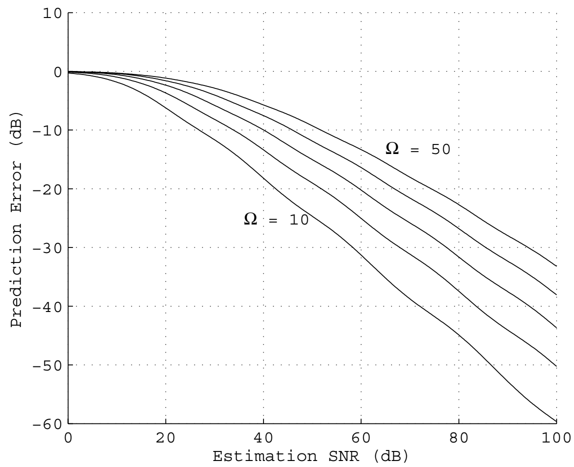

Fig. 1. Minimum mean squared prediction error of a Clarke-type fading envelope.

This allows us to evaluate $\phi_n(t)$ for t that are not quadrature points. Note that, in (23), only terms with significant eigenvalues need be used.

Also, from (20), note that K is not a symmetric matrix. Symmetry can be restored to the eigenvalue problem (22), though, using a straightforward technique [10, p. 794].

6. OBSERVATIONS AND CONCLUSIONS

Using Gaussian quadrature with $N = 96$ in (19) [11, p. 887], and making use of (17) and (23), we obtain the curves shown in Figure 1. These show the minimum mean squared prediction error of a unity-power Clarke-type mobile-radio fading envelope with $R_{xx}(t) = J_0(\Omega t)$ [4], where again, $x(t)$ is an Ω-bandlimited process. The prediction is based on past estimates over an interval of length $T = 0.2$ sec. The level of white-noise errors on this interval is expressed as a signal-to-noise ratio on the horizontal axis. The prediction is for $\tau = .04$ sec. in the future.

Note that, as the estimation SNR improves, the prediction error approaches zero. This is a general property of bandlimited processes [9]. Also, it is clear that the prediction error depends strongly on the maximum Doppler frequency $f_m = \Omega/2\pi$, but the curves in Figure 1 hardly vary from analogous ones plotted using (16). This suggests that the predictability of the mobile-radio fading envelope may be more profoundly affected by the maximum Doppler frequency than by the exact shape of the Doppler spectrum.

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