An Optimal Distributed and Adaptive Source Coding Strategy Using Rate-Compatible Punctured Convolutional Codes

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***Abstract*— A novel scheme exploiting rate-compatible punctured convolutional (RCPC) codes is proposed for *distributed* and *adaptive* source coding. The proposed scheme is based on the SF-ISF approach in [9]. It is simple, general, flexible, and provenly optimal. For the class of RCPC codes that are obtained from puncturing the parity bits of a recursive systematic convolutional (RSC) mother code, it is shown that an "optimal" codec exists where a single source encoder and a single source decoder can accommodate a set of different compression rates efficiently.**

I. Introduction

First stated in a simple seminal paper by Slepian and Wolf in 1973 [1], distributed source coding (DSC) has aroused considerable interest in the signal processing community in recent years. Through *separate* compressing but joint de-compressing of (correlated) sources, DSC promises the same overall compression rate as *joint* compressing. The technology is of particular interest to sensor networks, due to its ability to eliminate the redundancy within individual sensors as well as across sensors without explicit inter-sensor communication [2][3].

Unlike conventional compression methods, the key idea of DSC is to convert the source coding problem to an equivalent channel coding one and to exploit a linear channel code for the purpose of compression. Following the constructive demonstration in [2], a variety of efficient DSC schemes have been proposed using practical channel codes (eg. [2]-[9]). However, these existing schemes have primarily focused on *fixed-rate* compression, without addressing the flexibility issue. To fully harness the power of DSC, *rate adaptivity* is needed, which enables the sources (e.g sensor nodes) to continuously exploit the (time-varying) source correlation and compress at the just-right rate.

This paper investigates distributed and adaptive compression strategies that are capable of providing a set of different compression rates with efficiency and low complexity. Specifically, we propose a novel adaptive DSC scheme based on rate-compatible punctured convolutional (RCPC) codes. The scheme is simple, general, and provenly-optimal. To begin, RCPC codes are a family of convolutional codes with a set of distinct rates obtained from puncturing of the same mother code. The advantage of rate-compatible (RC) codes is that, since codewords of the high-rate codes in the family are embedded in those of the low-rate ones, a *single* pair of encoder and decoder is needed to perform channel coding for the entire family. When used as a source code, since the compression rate R_c is the complimentary of the code rate R: $R_c = 1 - R$, a set of distinct compression rates can be obtained. However, to optimally and efficiently exploit the different compression rates promised by the RC code (i.e. without rate loss and using minimal hardware) is not easy, since a DSC formulation tends to be strictly "customized" for a specific code, making it very difficult to concurrently accommodate multiple codes.

The novel strategy proposed here is based on the SF-ISF approach, a general DSC framework proposed in [7][9] that enables an arbitrary linear channel code to be optimally converted to a Slepian-Wolf code, provided that the syndrome former (SF) and the inverse syndrome former (ISF) of the channel code can be constructed. Since for most channels' codes, SF-ISF construction is straight-forward and a much lighter task than directly constructing the source encoder and source decoder, the SF-ISF framework has considerably simplified the DSC problem. Furthermore, for linear block codes, convolutional codes and parallelly/serially concatenated codes, systematic ways to construct SF-ISF pairs have been proposed [9]. This forms the base approach on which we build the adaptive scheme.

The key contribution of this paper is the extension of the SF-ISF construction proposed in [7][9], which works for non-punctured codes only, to punctured codes, and subsequently to RCPC codes. Specifically, we show that for RCPC families that are obtained from puncturing the parity bits of a recursive systematic convolutional (RSC) code, there exists a *single* pair of SF and ISF, implemented in simple linear sequential circuits, that work for all (punctured) codes in the RC family. This in turns enables the construction of one simple DSC codec with minimal hardware.

Finally, we note that the proposed strategy can be parallelly applied to rate-compatible punctured turbo (RCPT) codes using the SF-ISF formulation in [7][9]. This would

This material is in part supported by the National Science Foundation under Grant No. CCF-0430634 and by the Commonwealth of Pennsylvania, Department of Community and Economic Development, through the Pennsylvania Infrastructure Technology Alliance (PITA).

subsequently result in adaptive RCPT DSC scheme that is expected to have an even stronger compression capability (since turbo codes are stronger than convolutional codes), but will also have a higher complexity.

II. The Base Approach for DSC

The fundamental idea behind the proposed DSC scheme as well as the general SF-ISF approach is *code binning* [1]. At its information-theoretic root, code binning refers to a *distributed* strategy of using $2^{nH(X,Y)}$ (typical) sequences to describe a joint source (X, Y). The idea is to place the $2^{nH(X,Y)}$ sequences uniformly in $2^{nH(X|Y)}$ bins, and to assign $nH(X|Y)$ bits and $nH(Y)$ bits, respectively, for indexing the bins and enumerating specific sequences in a bin. (Here H denotes the Shannon entropy function.) In practice, code binning is applied to asymmetric DSC of binary i.i.d sources, where one source, say Y^n, is compressed using a conventional method (at rate $1:H(Y)$), and the other source, X^n, is compressed using an (n, k) linear channel code (at rate $n:(n-k)$) with the understanding that X and Y are correlated by $p = H(X \neq Y)$ and that Y will be losslessly available at the decoder to help recover X^n. Specifically, the compression of X^n can be performed through arranging the 2^n source sequences in the 2^{n-k} cosets as defined by the (n, k) channel code, and representing a source sequence (length n) using its coset index, or the *syndrome* (length $(n-k)$). To ensure the lossless and unique decodability, two basic conditions need to be satisfied: (1) The compression rate $\frac{n-k}{n}$ needs to be greater than or equal to the residual entropy $H(X|Y) = H(p)$, and (2) the (n, k) channel code needs to be capable of supporting reliable transmission of k information bits on BSC(p) channels [1].

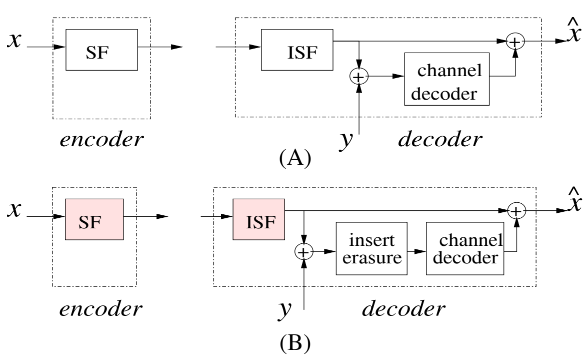


Fig. 1. (A): The universal SF-ISF codec proposed in [9]. (B). The modified SF-ISF codec for use with punctured codes and RCPC codes.

Note that the binning idea only outlines the construction of an optimal codebook, i.e. a triple of functions: $f_x : \mathcal{X} \rightarrow \mathcal{W}, f_y : \mathcal{Y} \rightarrow \mathcal{V}$, and $g : (\mathcal{W}, \mathcal{V}) \rightarrow (\mathcal{X}, \mathcal{Y})$, without illuminating the implementation of the codec. Hence, the efficiency with which compression and decompression can be performed depends on the actual realization of the DSC codec. For the base approach to construct adaptive DSC schemes, we turn to the SF-ISF codec [7][9], which is the only universal DSC codec that works optimally for a variety of practical channel codes. The structure of the SF-ISF codec is illustrated in Fig. 1(A), where the key components include the channel decoder (that comes with the channel code), and a pair of valid syndrome former and inverse syndrome former. Detailed discussion on the validity and optimality of the SF-ISF codec can be found in [9]. Here we wish to emphasize that this SF-ISF codec is particularly attractive for use in conjunction with rate compatible codes, since only a *single* channel decoder (that corresponds to the mother code) is needed in the source decoder.

III. Adaptive DSC Using RCPC Codes

To make use of the universal codec in Fig. 1, we need to construct SFs and ISFs for the family of RCPC codes. We start with the non-punctured code (i.e. the mother code), and then move on to the punctured code(s).

A. SF-ISF for Non-Punctured Convolutional Codes

As the name suggests, the role of the SF is to find the syndrome sequence for the given source sequence (virtual codeword of the channel code); and the role of the ISF is to find an *arbitrary* source sequence that is associated with the given syndrome sequence. While a table-lookup is always possible in theory, the complexity involved makes it impractical in real systems. The simple SF-ISF construction presented here is based on linear sequential circuits [7][9].

Let G denote the *generator matrix* of a rate k/n binary non-punctured convolutional code, which is formed from $k \times n$ generator polynomials in the $\mathcal{D}$-domain. Define the *transfer matrix*, H^T, as a $n \times (n-k)$ matrix in the $\mathcal{D}$-domain which has a rank $(n-k)$ and satisfies

$$GH^T = \mathbf{0}, \qquad (1)$$

where superscript T stands for matrix transposition. It can then be verified that H^T and its left inverse, $(H^{-1})^T$, form a valid pair of SF and ISF [7][9]. By left inverse, we mean

$$(H^{-1})^T H^T = I, \qquad (2)$$

where I is the identity matrix.

B. SF-ISF for Punctured Convolutional Codes

The above SF-ISF construction is simple and efficient, but works for non-punctured convolutional codes only, and hence, excludes a considerable set of candidate codes. To exploit potentially good punctured convolutional codes and particularly rate-compatible punctured convolutional codes, we propose to convert them to non-punctured equivalencies and subsequently construct the corresponding SF and ISF. Intuitively, such a transformation is possible (e.g. via polyphase transform) due to the linearity of the punctured code and the periodic relation between the source bits and the code bits. Rigorous proof will follow in the succeeding subsections.

Let G denote the $k \times n$ generator matrix of the rate $R_0 = k/n$ mother code and t denote the puncturing period. The equivalent non-punctured generator matrix of the punctured code can be obtained by first transforming G to G_t, an equivalent matrix of dimensionality $kt \times nt$, and then extracting the irrelevant columns. The key problem here is how to efficiently expand an arbitrary generator matrix G by a factor of t both in rows and columns.

Before we proceed to the main results of this paper, let us first discuss the notation. Note that the binary vector space and the $\mathcal{D}$-space has a one-to-one correspondence. A $\mathcal{D}$-domain non-recursive (i.e. feed-forward) polynomial $A(D) = \sum_{i=0}^{m} a_i D^i$ corresponds to a binary sequence $\bar{a} = [a_0, a_1, \cdots, a_m]$, where D is viewed as a delay element. The sequence $\bar{a}$ can be divided into t sub sequences according to the modulo-t positions, i.e.,

$$\begin{aligned} \bar{a}_0^{(t)} &= [a_0, a_t, a_{2t}, \cdots], \\ \bar{a}_1^{(t)} &= [a_1, a_{t+1}, a_{2t+1}, \cdots], \\ \cdots & \quad \cdots \\ \bar{a}_{t-1}^{(t)} &= [a_{t-1}, a_{2t-1}, a_{3t-1}, \cdots], \end{aligned}$$

whose $\mathcal{D}$-domain representations are given by

$$A_j^{(t)}(D) = \sum_{i=0}^{m} a_{it+j} D^i, \quad j = 0, 1, \cdots, t-1. \quad (3)$$

Clearly, we have

$$A(D) = A_0(D^t) + DA_1(D^t) + \cdots + D^{t-1} A_{t-1}(D^t). \quad (4)$$

The superscript (t) is omitted where there is no confusion.

To ease proposition, define $\tilde{A}^{(t)}(D) \triangleq$

$$\begin{bmatrix} A_0(D) & A_1(D) & \cdots & A_{t-2}(D) & A_{t-1}(D) \\ DA_{t-1}(D) & A_0(D) & \cdots & A_{t-3}(D) & A_{t-2}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ DA_1(D) & DA_2(D) & \cdots & DA_{t-1}(D) & A_0(D) \end{bmatrix},$$

as the *t-cyclic elementary matrix* of $A(D)$. Note that $A(D)$ is a *feed-forward* polynomial in the $\mathcal{D}$ domain, and $\tilde{A}^{(t)}(D)$ is a $t \times t$ square matrix that contains t^2 $\mathcal{D}$-domain feed-forward polynomials. We have the following properties for $\tilde{A}^{(t)}(D)$:

Lemma 1: (Properties of a t-cyclic elementary matrix)

- $\tilde{A}(D)$ is a full rank matrix provided that $A(D) \neq 0$.
- When $A(D) = 1$, $\tilde{A}(D)$ is an identity matrix.

Theorem 1: For a rate k/n convolutional code with generator matrix $G \triangleq [G_{i,j}]_{k \times n}$, where each entry $G_{i,j}(D) = U_{i,j}(D)/V_{i,j}(D)$, and $U_{i,j}(D)$ and $V_{i,j}(D)$ are feed-forward polynomials, the equivalent generator matrix G_t can be obtained by replacing each entry $G_{i,j}(D)$, with a $t \times t$ square matrix $\tilde{U}_{i,j}(D)(\tilde{V}_{i,j}(D))^{-1}$.

Proof: The proof of Lemma 1 and the Theorem 1 requires careful algebra, but is otherwise straight-forward. Hence it is omitted.

We note that the transformation from G to G_t is always valid since $\tilde{V}_{i,j}(D)$ is a full rank square matrix and hence its inverse always exists. Further, the Theorem is applicable to both systematic and non-systematic, recursive and non-recursive convolutional codes. For example, when the code is a non-recursive convolutional code, then $V_{i,j}(D) = 1$ and $\tilde{V}_{i,j}(D) = I_k$ for $\forall\ 1 \leq i \leq k$ and $1 \leq j \leq n$.

C. Adaptive DSC Using RCPC Codes

If we can represent any punctured convolutional code in an equivalent non-punctured closed form, then the SF-ISF construction method discussed in the previous sub Section can certainly be applied. It should be noted, however, in order for one channel decoder (that of the mother code) to accommodate the outputs from possibly different ISFs (see the decoder structure in Fig. 1(A)), erasures need to be inserted to align the sequences. This leads to the slightly modified source decoder structure which is shown in Fig. 1(B).

To summarize, exploiting RCPC codes in the SF-ISF codec for adaptive-rate DSC can take the following steps: (1) Transform the generator matrix of the mother code G to G_t using Theorem 1; (2) For each RCPC component code, delete the corresponding column(s) in G_t to form the closed-form generator matrix and construct the respective SF-ISF pair using (1) and (2); (3) *Combine* these SFs and ISFs in as few linear sequential circuits as possible, and insert them in the source encoder and the source decoder in Fig. 1(B); (4) Finally, insert the channel decoder of the mother code in the source decoder. This modular is invariant regardless of what (punctured) code (from the RCPC family) is used, provided that erasures are inserted properly.

Clearly, consolidating the set of SFs and ISFs is critical in reducing the complexity and size of the source encoder and the source decoder. If the set of SFs and ISFs can be completely packed into a single SF and a single ISF (the best case), then we say that the resulting DSC codec is "optimized" in complexity. The theorem below states that certain RCPC families are guaranteed to have an "optimized" DSC codec.

Lemma 2: For a rate k/n recursive systematic convolutional (RSC) code with generator matrix $G = [I_k, P_{k \times (n-k)}]$, a valid pair of SF and ISF can take the following form:

$$H^T = \begin{bmatrix} P_{k \times (n-k)} \\ I_{n-k} \end{bmatrix}, \quad (H^{-1})^T = \begin{bmatrix} \mathbf{0}_{(n-k) \times k}, I_k \end{bmatrix} \quad (5)$$

Theorem 2: For a family of RCPC codes obtained from puncturing the parity bits of a rate k/n mother RSC code using period t puncturing patterns, there exists an "optimized" DSC codec, where a single pair of SF and ISF that are derived from the $tk \times tn$ expanded generator matrix of the mother code using Lemma 2, work for all codes in the RCPC family.

Proof: Let G and G_t be the mother generator matrix and its expanded representation. Since G_t is an RSC generator matrix in the form of $G_t = [I_{kt}, P_{kt\times(nt\text{–}kt)}]$, a valid SF-ISF pair can be derived using Lemma 2, denoted as ${H_t}^T$ and $(H_t^{-1})^T$. Any punctured code in this RCPC family obtains the equivalent closed-form generator matrix by deleting certain column(s) in $P_{kt\times(nt\text{–}kt)}]$, which is systematic and a sub matrix of G_t. Hence, if constructed using Lemma 2, the SF and ISF for this punctured code are embedded in ${H_t}^T$ and $(H_t^{-1})^T$. In other words, ${H_t}^T$ and $(H_t^{-1})^T$ can fulfill the role of SF and ISF for this punctured code by switching off certain parts of the linear sequential circuits.

We not that Lemma 2 and Theorem 2 actually hold for systematic, but not necessarily recursive, convolutional codes. However, in practice, systematic but non-recursive convolutional codes tend to perform worse than RSC codes or non-systematic non-recursive codes.

Example: Consider a rate 1/2 RSC code with generator matrix $[1, \frac{1+D+D^2}{1+D^2}]$. Using a puncturing pattern $[1,1;1,0]$, a rate 3/4 punctured code can be ordained. Hence the RCPC family consists of two codes with rate 3/4 and 2/4 (the mother code), respectively, which, if used in adaptive DSC, can offer two compression rates of $4\!:\!1$ and $4\!:\!2$.

Let $U(D) \triangleq 1 + D + D^2$ and $V(D) \triangleq 1+D^2$, whose 2-cyclic elementary matrix are

$$\tilde{U}(D)=\begin{bmatrix} 1+D & 1 \\ D & 1+D \end{bmatrix}, \quad \tilde{V}(D)=\begin{bmatrix} 1+D, & 0 \\ 0, & 1+D \end{bmatrix},$$

Since $\tilde{U}(D)(\tilde{V}(D))^{-1} = \begin{bmatrix} 1, & \frac{1}{1+D} \\ \frac{D}{1+D}, & 1 \end{bmatrix} \triangleq P_{2\times 2}$,

the expanded matrix of G is given by $G_t = [I_2, P_{2\times 2}]$, and a possible choice of SF-ISF pair is (using Lemma 2):

$$H_t^T = \begin{bmatrix} 1, & \frac{1}{1+D} \\ \frac{D}{1+D}, & 1 \\ 1, & 0 \\ 0, & 1 \end{bmatrix}, \quad (H_t^{-1})^T = \begin{bmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{bmatrix}.$$

For the rate 3/4 punctured code, the equivalent closed-form (i.e non-punctured) generator matrix is obtained by deleting the forth column of G_t, i.e. $\begin{bmatrix} 1, & 0, & 1 \\ 0, & 1, & \frac{D}{1+D} \end{bmatrix}$,

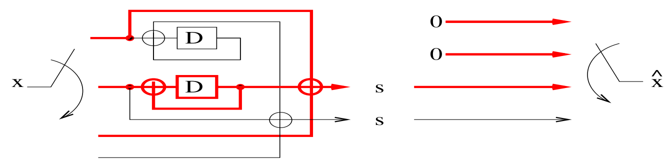
and a corresponding SF-ISF pair can be

$$H_1^T = \begin{bmatrix} 1, & \frac{D}{1+D}, & 1 \end{bmatrix}^T, \quad (H_1^{-1})^T = \begin{bmatrix} 0, & 0, & 1 \end{bmatrix}.$$

Clearly, H_1^T and $(H_1^{-1})^T$ are embedded in H_t^T and $(H_t^{-1})^T$. Fig. 2 shows the linear sequential circuits of H_t^T and $(H_t^{-1})^T$. Dark red lines also sketch the part that correspond to H_1^T and $(H_1^{-1})^T$.

D. Simulations of the Adaptive RCPC DSC Scheme

As an example, we evaluated a family of RCPC codes obtained from a rate $1/4$ mother code with generator matrix $[1, 171/133, 145/133, 127/133]_{oct}$. Homogeneous puncturing among the parity bits is performed which yields a series of compression rates $4\!:\!2$, $5\!:\!3$, $6\!:\!4$, $7\!:\!5$ and $8\!:\!6$. Fig. 3 shows the performance of these RCPC codes on memoryless binary symmetric sources (BSC), where the x-axis denotes the source correlation p and y-axis the normalized distortion. Clearly, while each code in the RCPC family is useful in its own way for fixed-rate DSC, they collectively can offer rate adaptivity to varying source correlations from $p \leq 0.027$, 0.047, 0.06, 0.08 to 0.10 (assuming a normalized distortion of 10^{-5} is near lossless).



(A) SF (B) ISF

Fig. 2. Linear sequential circuits for SF-ISF.

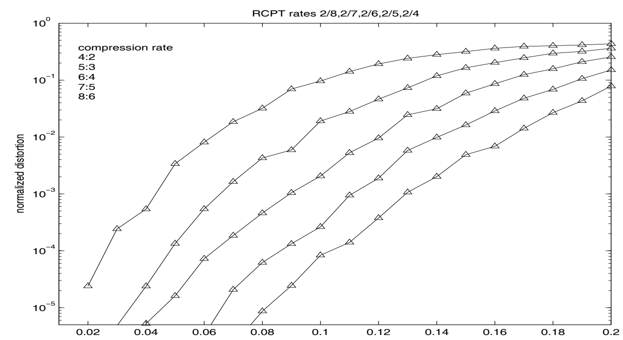


Fig. 3. Performance of the adaptive RCPC DSC scheme.

IV. Conclusion

We have proposed an optimal adaptive-rate distributed source coding scheme using RCPC codes. We show that a single DSC codec can be used to accommodate a set of distinct compression rates with minimal hardware. To the best of the authors' knowledge, this is the first adaptive DSC scheme exploiting rate-compatible linear codes.

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