SOURCE SEPARATION BY QUADRATIC CONTRAST FUNCTIONS: A BLIND APPROACH BASED ON ANY HIGHER-ORDER STATISTICS

Marc Castella[1], *Saloua Rhioui*[2], *Eric Moreau*[2] *and Jean-Christophe Pesquet*[1]

[1] Institut G. Monge and UMR-CNRS 8049
Univ. Marne la Vallée
F-77454 Marne la Vallée, Cedex 2, France
e-mail: {castellm,pesquet}@univ-mlv.fr

[2] STD, ISITV,
av. G. Pompidou, BP56,
F-83162 La Valette du Var, Cedex, France
e-mail: {rhioui,moreau}@univ-tln.fr

ABSTRACT

This paper deals with blind source separation by contrast function maximization. A general class of separation criteria valid for both i.i.d. and non i.i.d. sources is exhibited: it is based on third or higher order cross-cumulants between the separator outputs and fixed signals, called references. We show that this approach is applicable not only in a semi-blind context, but also in a completely blind scenario. The most interesting feature concerning our criteria is their *quadratic* form. It follows a highly simplified optimization procedure which is described in the paper. Simulation results illustrate the validity of our approach, and the appeal of this new class of contrast functions.

1. INTRODUCTION

We consider the blind equalization problem of multichannel Linear and Time Invariant (LTI) systems. This problem is also referred to as blind source separation (BSS) and occurs in many applications, in particular in multi-user digital communications. Solutions to this issue have been proposed in several works. The main explored directions include frequency point of views [1], global approaches [2, 3] which recover all the sources simultaneously, and iterative methods [4] which extract the sources one by one. This paper deals with the latter case, and more precisely with solutions which are based on the maximization of a criterion called contrast function. Such contrasts have been proposed for both i.i.d. [5] and non i.i.d. [6] source signals. Unfortunately, as they involve higher order statistics, their optimization appears to be computer intensive.

Recently, contrast functions have been generalized through the use of so-called reference signals. The first contributions have restricted themselves to the case of instantaneous source mixtures, [7, 8] while results concerning convolutive mixtures have been presented more recently [9]. Because of their optimization simplicity, these approaches are extremely appealing.

In this paper, our goal is to develop new quadratic contrast function for convolutive models. In particular we extend results in [9] to cumulants of any order greater than or equal to three. The validity of our contrast functions is established under general conditions, which brings more flexibility (possibility to choose freely some "reference" signals, to use complex conjugate or not). The usefulness of such a wide family of criteria is illustrated by computer simulations, where third order cumulants are shown to perform better for some classes of sources. In addition, it is proved both theoretically and by simulations that our criteria can be used in a completely blind context which constitutes an extension of our earlier results in [9].

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2. MODEL AND PROBLEM FORMULATION

We consider a Q-dimensional ($Q \in \mathbb{N}, Q \geq 2$) discrete-time signal, which is observed. It is denoted by the column vector $\mathbf{x}(n)$, where in the whole paper, n stands for any integer ($n \in \mathbb{Z}$). The observation $\mathbf{x}(n)$ results from a LTI multichannel system described by the input-output relation:

$$\mathbf{x}(n) = \sum_{k \in \mathbb{Z}} \mathbf{M}(k)\mathbf{s}(n-k). \qquad (1)$$

$\mathbf{M}(n)$ represents a sequence of (Q, N) matrices which corresponds to the impulse response of the LTI mixing system and $\mathbf{s}(n)$ is an N-dimensional ($N \in \mathbb{N}^*$) unknown and unobserved column vector, which is referred to as the vector of *sources*. All quantities may be either real or complex-valued.

As our approach is an iterative one, we will focus on the extraction of a single source. Hence, using only the observations $\mathbf{x}(n)$, the considered problem consists in estimating a $(1, Q)$ LTI vector filter, called equalizer and with impulse response $\mathbf{w}(n)$, such that the scalar signal

$$y(n) = \sum_{k \in \mathbb{Z}} \mathbf{w}(k)\mathbf{x}(n-k) \qquad (2)$$

restores one of the components $s_i(n), i \in \{1, \ldots, N\}$ of the source vector. More precisely, defining the $(1, N)$ global LTI vector filter $\mathbf{g}(n)$ by the following impulse response:

$$\mathbf{g}(n) = \sum_{k \in \mathbb{Z}} \mathbf{w}(k)\mathbf{M}(n-k) , \qquad (3)$$

we have

$$y(n) = \sum_{k \in \mathbb{Z}} \mathbf{g}(n-k)\mathbf{s}(k) \triangleq \{\mathbf{g}\}\mathbf{s}(n) . \qquad (4)$$

We say that the equalization is achieved when there exists an index $i_0 \in \{1, \ldots, N\}$ and a non-zero scalar filter with impulse response $g(n)$, such that the filter components in $\mathbf{g}(n)$ read

$$g_i(n) \triangleq (\mathbf{g}(n))_i = \alpha g(n)\delta_{i-i_0} , \qquad (5)$$

where $\alpha \in \mathbb{C}^*$ and δ_k stands for the Kronecker symbol, *i.e.* $\delta_k = 1$ if $k = 0$ and 0 otherwise. The above relation is called the "equalization condition" and expresses the fact that $y(n)$ corresponds to the source $s_{i_0}(n)$ up to a scalar filtering.

Notice that the above equalization criterion can be made more restrictive when the source signals are also assumed to be independent sequences of i.i.d. (independent and identically distributed) complex random variables. Indeed, in such a case, it is classically said that the equalization is realized when the scalar filter $g(n)$ reduces to a delay, which reads:

$$\exists l \in \mathbb{Z} \quad g_i(n) \triangleq (\mathbf{g}(n))_i = \alpha\delta_{n-l}\delta_{i-i_0} \,. \tag{6}$$

In order to be able to solve the BSS problem, we need to introduce some assumptions on the source signal. The following classical assumption is made, which is known to play a key role in these problems:

A.1 The source vector components $s_i(n), i \in \{1, \dots, N\}$ are *mutually independent*, stationary and zero-mean processes with unit variance. Their respective covariance function are positive definite functions denoted by $\gamma_i(k), k \in \mathbb{Z}$ for all $i \in \{1, \dots, N\}$.

Since the sources have unit variance, one can restrict the multiplicative factor in (5) and (6) to $|\alpha| = 1$ by imposing the constraint $\mathrm{E}\{|y(n)|^2\} = 1$. Equivalently, defining the norm of the global $(1, N)$ filter by

$$\|\mathbf{g}\|^2 \triangleq \sum_{i=1}^{N} \sum_{(k_1,k_2)\in\mathbb{Z}^2} g_i(k_1) g_i^*(k_2) \gamma_i(k_2 - k_1) \tag{7}$$

we can work within the set of unit-norm global filters ($\|\mathbf{g}\| = 1$). Note that for independent identically distributed sources, this condition reads: $\sum_{i=1}^{N} \sum_{k\in\mathbb{Z}} |g_i(k)|^2 = 1$.

3. MISO SEPARATION CRITERIA

3.1. Background

The concept of contrast function has been introduced in BSS so as to reduce the problem to an optimization one: by definition, a contrast function is a criterion which is maximum only if the equalization condition (5) (or (6) if the sources are i.i.d.) is satisfied.

Denoting by $\mathrm{C}_4\{.\}$ the fourth-order auto-cumulant of a random variable, it is known that the criterion $|\mathrm{C}_4\{y(n)\}|$ constitutes a contrast function in the case of both i.i.d. sources [5] and non i.i.d. sources [6]. Unfortunately, the optimization of the latter criterion requires an iterative gradient-like procedure which makes it computationally intensive.

3.2. A family of contrast functions

The first contribution of the paper consists in using criteria based on R-th order ($R \geq 3$) cross-cumulants, where $R-2$ variables are fixed. This choice yields a quadratic dependence with respect to the optimized parameter, which greatly simplifies the optimization task. More precisely, to give a general expression, we introduce the notation $\mathsf{y}(n)$ to designate either $y(n)$ or its complex conjugate $y^*(n)$. We then define the following R-th order ($R \geq 3$) cumulant:

$$\kappa_{R,z}\{\mathsf{y}(n)\} \;=\; \mathrm{Cum}\{\mathsf{y}(n), \mathsf{y}(n), z_1(n), \dots, z_{R-2}(n)\} \tag{8}$$

where $z_i(n), i \in \{1, \dots, R-2\}$ are given signals. In our previous work [9], they have been referred to as *reference* signals determined from prior information, but we will prove that they may be chosen in a rather arbitrary way. We now define the following criterion:

$$\mathcal{C}_{R,z}\{\mathsf{y}(n)\} \triangleq |\kappa_{R,z}\{\mathsf{y}(n)\}| \,, \qquad R \geq 3 \tag{9}$$

For the sake of clarity, we will focus on i.i.d. source signals although the proofs can be extended to the non i.i.d. case. We need to define the following supremum, where $\mathsf{s}_j(n-k)$ is conjugated in the same way as $\mathsf{y}(n)$ in (8) and (9):

$$\kappa_R^{\max} = \max_{j=1}^{N} \sup_{k\in\mathbb{Z}} |\kappa_{R,z}\{\mathsf{s}_j(n-k)\}| \tag{10}$$

The proof of Proposition 1 requires the following assumption, which will appear to be fulfilled subsequently:

A.2 There exists (j_0, l_0) such that:

$$\kappa_R^{\max} = |\kappa_{R,z}\{\mathsf{s}_{j_0}(n-l_0)\}| < +\infty \tag{11}$$

We can then state:

Proposition 1 *In the case of i.i.d. source signals, the criterion* $\mathcal{C}_{R,z}$ *is a contrast under unit norm constraint* ($\|\mathbf{g}\| = 1$) *if and only if the set*

$$\mathcal{I} \triangleq \{(j,k) \in \{1, \dots, N\} \times \mathbb{Z} \mid |\kappa_{R,z}\{\mathsf{s}_j(n-k)\}| = \kappa_R^{\max}\} \tag{12}$$

contains a single element.

Proof: For the sake of clarity, we will give the proof only for the criterion $\mathcal{C}_{R,z}\{y(n)\}$ derived from (8) where $\mathsf{y}(n) = y(n)$. It can be easily adapted to other cases. We can then write:

$$\kappa_{R,z}\{y(n)\} = \sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} g_j(k)^2 \kappa_{R,z}\{s_j(n-k)\} \tag{13}$$

and, using (10) and the unit-norm property of $\mathbf{g}(n)$, it follows

$$\mathcal{C}_{R,z}\{y(n)\} \leq \sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} |g_j(k)|^2 |\kappa_R\{s_j(n-k)\}| \tag{14}$$

$$\leq \kappa_R^{\max} \sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} |g_j(k)|^2 = \kappa_R^{\max} \,. \tag{15}$$

If the above upper-bound is reached (which is possible according to assumption A.2), then

$$\sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} |g_j(k)|^2 \;\; \kappa_R^{\max} - |\kappa_{R,z}\{s_j(n-k)\}| \;\; = 0 \tag{16}$$

All terms in the above sum being positive, if $\mathcal{I}$ contains a single element, one deduces, that the global Multi-Input / Single-Output filter $\{\mathbf{g}\}$ satisfies the equalization condition (6).

Conversely, one can see that if $\mathcal{I}$ contains several elements, there exist non separating filters which maximize $\mathcal{C}_{R,z}$. ■

3.3. Blind choice of $z_i(n)$

We have not specified so far how to choose the signals $z_i(n)$. The simplest way consists in assuming that each signal is obtained by a MISO finite impulse response (FIR) filtering of the sources:

$$\forall i \in \{1, \ldots, R-2\} \quad z_i(n) = \sum_{k \in \mathbb{Z}} \mathbf{t}^{(i)}(n-k)\mathbf{s}(k)\,. \qquad (17)$$

where for all i, $\mathbf{t}^{(i)}(n)$ is the impulse response of a $(1, Q)$ FIR (hence stable) vector filter. It follows from the stability of the filters $\{\mathbf{t}_i\}, i \in \{1, \ldots, R-2\}$ that *assumption A.2 is satisfied.* The following proposition ensures that $z_i(n), i \in \{1, \ldots, R-2\}$ can be chosen blindly, contrary to what has been done in [9]:

Proposition 2 *If the coefficients of the filters $\{\mathbf{t}_i\}$, $1 \leq i \leq R-2$ have been chosen randomly distributed according to a continuous joint probability density function (supported on a set of non-zero measure), then almost surely the set $\mathcal{I}$ has one single element and $\mathcal{C}_{R,z}$ is a contrast.*

<u>*Proof:*</u> Using cumulant multilinearity and source independence, one obtains (for any given j in $\{1, \ldots, N\}$):

$$|\kappa_{R,z}\{s_j(n-k)\}| = |\mathrm{C}_R\{s_j(n)\} \prod_{i=1}^{R-2} t_j^{(i)}(k)| \qquad (18)$$

where $t_j^{(i)}(k)$ is the j-th component of $\mathbf{t}^{(i)}(k)$.

Denote by (j_0, l_0) a couple of indices satisfying (11). From (18), one can see that for any $(j,k) \neq (j_0, l_0)$, we have $(j,k) \in \mathcal{I}$ if and only if

$$|\mathrm{C}_R\{s_j(n)\} \prod_{i=1}^{R-2} t_j^{(i)}(k)| = |\mathrm{C}_R\{s_{j_0}(n)\} \prod_{i=1}^{R-2} t_{j_0}^{(i)}(l_0)|\,, \qquad (19)$$

which is almost surely false if the coefficients are driven from a continuous joint probability density function. ■

Notice that in order to satisfy the assumption of Proposition 1, we must have $\kappa_R^{\max} > 0$. This implies a necessary assumption on the sources, which we assume to be satisfied throughout the paper:

A.3 There exists $j \in \{1, \ldots, N\}$ such that the R-th order auto-cumulant of the j-th source is non zero, that is: $\mathrm{C}_R\{s_j(n)\} \neq 0$

4. OPTIMIZATION METHOD

We now give some details concerning the optimization of the proposed contrast. We will assume from now on that the mixing filter admits a MIMO-FIR left inverse filter of length D, which can be assumed to be causal because of the delay ambiguity. The row vectors which define the impulse response can be stacked in the following $(1, QD)$ row vector:

$$\underline{\mathbf{w}} \triangleq \ \mathbf{w}(0) \quad \ldots \quad \mathbf{w}(D-1)) \qquad (20)$$

Defining also the $(QD, 1)$ column vector

$$\underline{\mathbf{x}}(n) \triangleq \ \mathbf{x}(n)^T \quad \mathbf{x}(n-1)^T \quad \ldots \quad \mathbf{x}(n-D+1)^T \ ^T \qquad (21)$$

and the covariance matrix $\mathbf{R} = \mathrm{E}\{\underline{\mathbf{x}}(n)\underline{\mathbf{x}}(n)^T\}$ we can write $y(n) = \underline{\mathbf{w}}\,\underline{\mathbf{x}}(n)$ and hence $\mathrm{E}\{|y(n)|^2\} = \underline{\mathbf{w}}\mathbf{R}\underline{\mathbf{w}}^H$. Finally, using cumulant multilinearity, one can easily define a matrix $\mathbf{C}$ such that $\kappa_{R,z}\{y(n)\} = \underline{\mathbf{w}}\mathbf{C}\underline{\mathbf{w}}^H$. The source separation task then amounts to the optimization problem:

$$\max |\underline{\mathbf{w}}\mathbf{C}\underline{\mathbf{w}}^H| \text{ under the constraint: } \underline{\mathbf{w}}\mathbf{R}\underline{\mathbf{w}}^H = 1 \qquad (22)$$

Now, for any row vector such that $\underline{\mathbf{w}}_0^H \in \ker \mathbf{R}$ we have $\underline{\mathbf{w}}_0\mathbf{R}\underline{\mathbf{w}}_0^H = \mathrm{E}\{\underline{\mathbf{w}}_0\mathbf{x}(n)\mathbf{x}(n)^H\underline{\mathbf{w}}_0^H\} = 0$ and hence the signal $\underline{\mathbf{w}}_0\mathbf{x}(n)$ vanishes identically. It follows that we may impose in addition $\underline{\mathbf{w}}^H \in (\ker \mathbf{R})^\perp$ to the optimization problem given by (22). By projection onto the subspace, one finally reduces the problem to the following one:

$$\max |\tilde{\mathbf{x}}^H\tilde{\mathbf{C}}\tilde{\mathbf{x}}| \text{ under the constraint: } \tilde{\mathbf{x}}^H\tilde{\mathbf{x}} = 1 \qquad (23)$$

which solution is known to be given by the singular vector of $\tilde{\mathbf{C}}$ corresponding to singular value with greatest modulus. Hence a SVD decomposition allows us to obtain the exact solution to the optimization problem.

5. SIMULATION RESULTS

Computer simulations are now presented to illustrate the usefulness of the above derivations. Two different types of real-valued source signals have been considered. The first one corresponds to i.i.d. signals taking their values in the set $\{-\frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}}\}$ with equal probabilities also known as PAM-4 and the second one to i.i.d. signals taking their values in the set $\{-1, 0, \alpha\}$ with the respective probabilities $\{\frac{1}{1+\alpha}, \frac{\alpha-1}{\alpha}, \frac{1}{\alpha(1+\alpha)}\}$ called MS(α) [10]. In the latter case, the parameter α is such that $\alpha \geq 1$ and it provides an easy way to parametrize the values of higher order cumulants.

In all experiments, we have taken $N = 3$ source signals, which have been mixed using a $(4, 3)$ FIR matrix filter of length 3. Hence the number of observed signals is $Q = 4$. The mixing filter coefficients have systematically been randomly chosen according to a normal distribution.

The reference signals $z_i(n), i \in \{1, \ldots, R-2\}$ have all been chosen equal: $\forall i, z_i(n) = z(n)$, where $z(n)$ is the output of a $(1, Q)$ FIR row filter of length 3 operating on the observed signals. The coefficients of this filter have also been chosen randomly using a normal distribution.

Based on the result of Proposition 1, one can estimate a first source signal $y(n)$ based on the former choice ($\forall i, z_i(n) = z(n)$) and the method described in Section 4. One can then set $\forall i, z_i(n) = y(n)$ and repeat the same procedure. When this procedure is repeated iteratively, the iteration number is denoted by N_i.

In all cases, the mean square estimation errors (MSE) of the sources have been evaluated over 100 Monte-Carlo runs. In each run, the source signals, the mixing system and the initial reference system have been randomly chosen.

Experiment 1: *About iteration number*

For PAM-4 source signals, we have estimated the average MSE versus the iteration number N_i. Notice that when $N_i = 1$, only one randomly (hence blindly) chosen reference system is involved. In this experiment, a fourth order cumulant contrast has been considered with a number of samples set successively to $K_1 = 10000$ and $K_2 = 50000$. The results are reported in the table below.

N_i	1	2	3	4	5
K_1	0.168	0.0046	$3\ 10^{-4}$	$2.6\ 10^{-4}$	$2.6\ 10^{-4}$
K_2	0.04	$1.42\ 10^{-4}$	$5.64\ 10^{-5}$	$5.62\ 10^{-5}$	$5.62\ 10^{-5}$

One can remark that even when $N_i = 1$, good estimation performance is obtained when the number of samples is large enough.

A more careful study has illustrated that with fewer samples, unsatisfactory results are due to estimation errors. In this case, a few iterations are required to obtain a good and constant performance level.

Experiment 2: *About data number*

For MS(α) source signals, with $\alpha = 3$ and $\alpha = 1+\sqrt{2}$, we have plotted in Figure 1 the average MSE versus number of samples. We have used $N_i = 3$ iterations. Note that the respective values of third and fourth order cumulants are $\mathrm{C}_3\{s(n)\} = 2$ $\mathrm{C}_4\{s(n)\} = 4$ for $\alpha = 3$ and $\mathrm{C}_3\{s(n)\} = \mathrm{C}_4\{s(n)\} = \sqrt{2}$ for $\alpha = 1+\sqrt{2}$. We have compared the performance obtained when using a third order cumulant contrast and a fourth order cumulant one. Better estimation performance has been obtained using a third order cumulant contrast. Hence one can take advantage of the skewness of the source probability density function by using a third order cumulant contrast.

Experiment 3: *MIMO case*

Resorting to a deflation procedure [4], we have considered the estimation of 3 sources. The average MSE for the three estimated sources (denoted respectively by s^1, s^2 and s^3 hereafter) versus the number of samples K is reported in the following table.

K	5000	10000	25000	50000
s^1	0.0008	0.0003	0.0001	$0.0659\ 10^{-3}$
s^2	0.0050	0.0021	0.0010	$0.4209\ 10^{-3}$
s^3	0.0064	0.0033	0.0018	$0.6277\ 10^{-3}$

As classically observed the estimation performance is better for the first estimated source signal.

Experiment 4: *Complex signals*

In this last experiment, we have considered the case of complex QAM-4 source signals. Using $K = 3000$ samples, we have plotted in Figure 2 one observed signals in the complex plane and one estimated source signals using an iteration number $N_i = 3$. One can notice that our method provides good results in the recovery of complex-valued communication signals. This has also been confirmed by a Monte Carlo study.

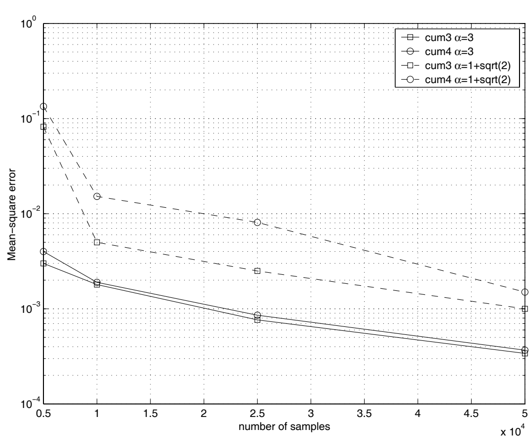

Fig. 1. For MS(α), average MSE versus number of samples considering contrasts based on third and fourth order cumulants.

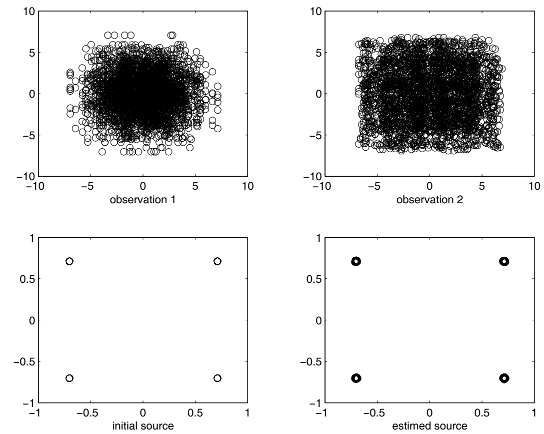

Fig. 2. For QAM-4 source signals, constellations of the observed signals, the estimated source signal and the corresponding source.

6. REFERENCES

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