PERFORMANCE AND CONVERGENCE ANALYSIS OF JOINT SOURCE-CHANNEL TURBO SCHEMES WITH VARIABLE LENGTH CODES

Xavier Jaspar and Luc Vandendorpe

Commun. and Remote Sensing Lab. — Université catholique de Louvain (UCL)
Place du Levant, 2 — B-1348 Louvain-la-Neuve, Belgium
Email: {jaspar,vandendorpe}@tele.ucl.ac.be

ABSTRACT

Recently, we proposed in [8, 9] a joint source-channel (JSC) turbo-(de)coder combining a variable length code (VLC) and a turbo-code, and applied it to image transfer. The simulation results showed the better performance and the lower decoding complexity of this scheme over the previous ones.

The goal of this paper is to assess the performance of such a scheme at all signal to noise ratios on the channel. Analytical tools are provided for the understanding and optimization of a wide variety of similar schemes currently explored in the literature. More precisely, the bit, symbol and frame distance spectra are developed for VLCs and for VLCs turbo-concatenated with an error correcting code (ECC), in order to get bounds on the corresponding error rates. Also EXIT charts are extended to three dimensions in order to analyze the turbo-convergence of the scheme.

1. INTRODUCTION

Joint source-channel (JSC) turbo-(de)coding techniques, based on variable length codes (VLCs), were recently developed in several ways [2, 1, 7, 10], focusing essentially on the VLC or on the concatenation with an error correcting code (ECC). Because these schemes were still less powerful than turbo-codes, improvements [6,11] were also proposed for the ECC part by using a turbo-code, however resulting in a decoder of high complexity if optimal — the meaning of optimal is given in section 2.

Pushing the use of a turbo-code further, we proposed in [9] a turbo-decoder, optimal *and* of low complexity, with three soft-in/soft-out (SISO) modules for a VLC concatenated with a turbo-code. Better performances were obtained, compared to previous schemes and to a classic turbo-code. Some design issues were discussed, essentially based on simulations results.

This is why *in this paper*, we provide rigorous tools for the analysis and the performance prediction of this new scheme. These tools include the distance spectra and union bounds on the error rates, as well as EXIT charts [12] extended to three dimensions for turbo-convergence analysis. Undoubtedly they give for future works the interesting possibility of analyzing the schemes cited above and comparing them to each other.

Xavier Jaspar thanks the Belgian F.N.R.S. for their financial support. This work is partly funded by the Federal Office for Scientific, Technical and Cultural Affairs, Belgium, through IAP contract No P5/11.

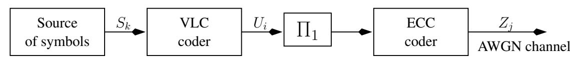

Fig. 1. Generic coder used throughout the paper.

2. CODER AND DECODER

The coder is depicted in Fig. 1. All previous works cited in the introduction use such a coder, except the parallel concatenation of [10] and the absence of the interleaver Π_1 in [6, 11]. For all those schemes, the tools developed in this paper are valid — or can be extended for [6, 11, 10].

The turbo-decoder can be deduced from the application of the sum-product algorithm onto the factor graph corresponding to Fig. 1. The decoder is considered optimal if the factor graph has no short cycles, since its only suboptimality then resides in the limited interleaver(s). See [8] and references therein for details.

Note that the primary purpose of the interleaver Π_1 is to break the decoding complexity. The next section will point out that this interleaver can also offer *an additional performance gain*.

3. DERIVATION OF THE BOUNDS

It is well known that the bit, symbol and frame error rates (resp. BER, SER, FER) are minimized with respectively the bit-, symbol- and frame-MAP (maximum-a-posteriori) detections. To the frame-MAP detection corresponds the suboptimal frame-ML (maximum-likelihood) detection. The proposed bounds are based on the *union bound* which is valid for the frame-ML detection from medium to high SNRs (signal to noise ratios on the channel) and remains valid in practice [3] for turbo-decoders — which are actually suboptimal symbol-MAP decoders.

3.1. Background on distance spectra

Let us now recall a few principles and notations about distance spectra, see [4] for details. Given a linear block code C, let $A^C_{w,h}$ (resp. $A^C_{w,j}$) denote the number of codewords with weight h (resp. parity check weight j) generated by information words of weight w. For convolutional codes, the same notations hold in terms of "paths" instead of "codewords". The *input-redundancy* (IRWEF) and *input-output weight enumerating function* (IOWEF) are defined respectively as $A^C(W,Z) \triangleq \sum_{w,j} A^C_{w,j}\, W^w Z^j$ and $A^C(W,H) \triangleq \sum_{w,h} A^C_{w,h}\, W^w H^h$.

The union bounds on the FER and BER, for a linear block code C with N_b information bits, are respectively:

$$\begin{aligned} P(e) &\triangleq \textstyle\sum_{w,h} A^C_{w,h}\, P_h, \\ P_b(e) &\triangleq \textstyle\sum_{w,h} \frac{w}{N_b} A^C_{w,h}\, P_h. \end{aligned} \tag{1}$$

where $P_h = \frac{1}{2}\,\mathrm{erfc}(\sqrt{hR_cE_b/N_0})$ for an AWGN channel with BPSK modulation; R_c is the code rate; N_0 is the one-sided noise psd and E_b the bit energy. For evaluating the bounds (1), we need the IOWEF. We refer the reader to the abstract concept of the *uniform interleaver* and to the methods proposed by Benedetto et al. in [4] for computing the IOWEF of concatenated linear codes.

However these methods can not be applied here because of the VLC non linearity. One main contribution of this paper is to develop and demonstrate similar methods — to compute the IOWEF and the bounds — for VLCs concatenated with linear codes.

3.2. Mean spectra of VLCs and bounds

Only a summary of the results and developments is presented due to the lack of place. Also, for the sake of clarity, the following assumptions are made: the source symbols are independent; the source code is a Huffman-like VLC; the number of bits or symbols is fixed, not both.

For a VLC, we are interested essentially in the union bounds on the BER and the SER. We start with the SER case which is simpler. The aim is to get the SER for a symbol S_i, i.e.:

$$\mathrm{SER} \triangleq \mathbf{E}\{\mathbb{I}(\check{S}_i \neq S_i)\} = P(\check{S}_i \neq S_i), \tag{2}$$

where $\mathbf{E}\{.\}$ denotes an expectation, $\mathbb{I}(.)$ is an indicative function ($\mathbb{I}(a) = 1$ if a is true, 0 otherwise), S_i is the emitted source symbol and $\check{S}_i$ the decision at the receiver. In the sequel, a symbol trellis is considered and the symbol S_i is produced during the transition between the trellis positions $i-1$ and i. Then, the expression $\check{S}_i \neq S_i$ means an error event $\underline{\mathbf{E}}^{l_s}_{i,i-k}$, of l_s symbols long, begins at the trellis position $i-k$, with at least one symbol error in S_i:

$$\mathrm{SER} \leq \sum_{\underline{\mathbf{s}}} \sum_{l_s=1}^{\infty} \sum_{k=1}^{l_s} P(\underline{\mathbf{E}}^{l_s}_{i,i-k}|\underline{\mathbf{s}})\, P(\underline{\mathbf{s}}), \tag{3}$$

where $\sum_{\underline{\mathbf{s}}}$ denotes a summation over all possible emitted sequences of symbols (thus including the symbol S_i). Note the simplified notation $P(\underline{\mathbf{S}} = \underline{\mathbf{s}}) = P(\underline{\mathbf{s}})$ is used.

By further developing $P(\underline{\mathbf{E}}^{l_s}_{i,i-k}|\underline{\mathbf{s}})$, the *bound on the SER* for VLCs is eventually obtained as:

$$\mathrm{SER} \leq \sum_{h\geq1} P_h \sum_{s\geq1} s\, A^{\mathrm{VLC}}_{s,h} \tag{4}$$

$$A^{\mathrm{VLC}}_{s,h} \triangleq \sum_{\underline{\mathbf{s}}_0} P(\underline{\mathbf{s}}_0) \left|\left\{ \underline{\mathbf{s}'}_0 \middle|^{d_H(\underline{\mathbf{s}}_0,\underline{\mathbf{s}'}_0)=h,\ d_S(\underline{\mathbf{s}}_0,\underline{\mathbf{s}'}_0)=s,}_{\text{first remerge at } l(\underline{\mathbf{s}}_0)=l(\underline{\mathbf{s}'}_0)} \right\}\right| \tag{5}$$

where $A^{\mathrm{VLC}}_{s,h}$ is called the *mean symbol-spectrum*; $|\{.\}|$ is the number of elements in the ensemble $\{.\}$; $\underline{\mathbf{s}}_0$ and $\underline{\mathbf{s}'}_0$ are sequences diverging from each other at time 0, and merging back for the first at time $l(\underline{\mathbf{s}}_0)$; $d_H(a,b)$ (resp. $d_S(a,b)$) is the hamming (resp. symbol) distance between a and b; $l(a)$ is the bit length of a; and P_h is defined in (1).

The distance $d_S(a,b)$ can be also the Levenshtein distance [2] between a and b. The SER is then denoted $\mathrm{SER_L}$. The *free distance* d^{VLC}_f of the VLC is defined as the minimum h such that $\sum_{s\geq1} A^{\mathrm{VLC}}_{s,h} \neq 0$.

A relation similar to (4) was firstly proposed in the pioneering results of [5]. The originality here resides in the more general and rigorous development used above to get it. Furthermore, this particular development can be and is extended to provide below the bit-spectrum of VLCs and in section 3.3 the spectra of block-VLCs — for the first time in the literature, to our knowledge.

The second error rate of interest is the BER. Even though it is not fully informative w.r.t. the source distortion, a bound on the BER gives the interesting possibility of being able to compare the performance of a VLC against the one of an ECC.

The BER on a bit U_i is defined as $\mathrm{BER} \triangleq \mathbf{E}\{\mathbb{I}(\check{U}_i \neq U_i)\}$. A bit trellis is now considered. The only difference w.r.t. the development of the SER is that the probability for an error event to begin at a given trellis position is now function of the probability to be in the root state (see [8] for an explanation of the root state) at that position. Consequently, the term $P(\underline{\mathbf{E}}^{l_b}_{i,i-k}|\underline{\mathbf{u}})$ similar to (3) can be expanded as:

$$P(\underline{\mathbf{E}}^{l_b}_{i,i-k}|\underline{\mathbf{u}})P(\underline{\mathbf{u}}) = P(\underline{\mathbf{E}}^{l_b}_{i,i-k}|\underline{\mathbf{u}},\mathbf{R}_{i-k})\, P(\underline{\mathbf{u}}|\mathbf{R}_{i-k})\, P(\mathbf{R}_{i-k})$$

It can be demonstrated that the probability $P(\mathbf{R}_k)$ to be in the root state in k is given by $P(\mathbf{R}_k) = (1/\bar{l})\ \mathrm{HCF}(\{l_i\})$, where $\mathrm{HCF}(\{l_i\})$ is the highest common factor of the VLC lengths l_i, and $\bar{l}$ the average length. Eventually, the BER is bounded by:

$$\mathrm{BER} \leq \sum_{h\geq1} P_h \sum_{w\geq1} w\ A^{\mathrm{VLC}}_{w,h}. \tag{6}$$

We call $A^{\mathrm{VLC}}_{w=h,h} = (1/\bar{l})\ \mathrm{HCF}(\{l_i\}) \sum_{s\geq1} A^{\mathrm{VLC}}_{s,h}$ the *mean bit-spectrum*. Note we can simplify (6) because $A^{\mathrm{VLC}}_{w,h} = 0$ for $w \neq h$.

If the HCF is bigger than 1, the trellis is stationary with a periodicity and the BER in (6) must then be considered as an average over the period.

Finally, the well known *event error rate* can also be bounded:

$$\mathrm{EER} \leq \sum_{h\geq1} P_h \sum_{w\geq1} A^{\mathrm{VLC}}_{w,h}. \tag{7}$$

This result is different from [5].

3.3. Mean spectra of block-VLCs

We call *block-VLC* a VLC limited to a frame. Obviously, there is a link between the spectra of a VLC and the one of the corresponding block-VLC. In [3], a method to get the block version spectra is given for linear CCs. Here, without entering into details, a simple transformation is given in the polynomial formalism, which is valid for both linear CCs and non linear VLCs. For VLCs, two variants of it exist because both a symbol- and a bit-clock can be used to concatenate the error events in the transformation.

The *symbol-clock transformation*, for example, of the mean symbol-spectrum of the VLC into the one of the block-VLC (b-VLC) can be written with these original polynomial relations:

$$\begin{aligned} A^{\text{b-VLC}}_{s,h} &= \textstyle\sum_{l_s,n\geq1} \binom{N_s-l_s+n}{n}\, T^{\mathrm{VLC}}_{s,h,l_s,n}, \\ T^{\mathrm{VLC}}(S,H,L_s,\Omega) &= \textstyle\sum_{s,h,l_s,n} T^{\mathrm{VLC}}_{s,h,l_s,n}\, S^s H^h L_s^{l_s} \Omega^n, \\ &= \textstyle\sum_{a=1}^{\infty} \left(A^{\mathrm{VLC}}(S,H,L_s)\,\Omega\right)^a, \\ A^{\mathrm{VLC}}(S,H,L_s) &= \textstyle\sum_{s,h,l_s} A^{\mathrm{VLC}}_{s,h,l_s} S^s H^h L_s^{l_s}, \end{aligned}$$

where N_s is the number of symbols in the block-VLC; $T^{\mathrm{VLC}}_{s,h,l_s,n}$ is implicitly defined; and $A^{\mathrm{VLC}}_{s,h,l_s}$ is an obvious extension of (5) with l_s the symbol length of the error events.

Table 1. Symbols probabilities and VLCs used.

	Prob.	VLC	$\mathrm{RVLC}_{d_f=1}$	$\mathrm{RVLC}_{d_f=2}$
	0.33	11	00	10
	0.30	10	01	01
	0.18	00	10	000
	0.10	011	111	111
	0.09	010	11011	1100
Entropy	2.139	-	-	-
Avg. length $\bar{l}$	-	2.19	2.37	2.46
Relative code rate	-	1.0	0.9241	0.8902
Free dist. d_f^{VLC}	-	1	1	2

Table 2. Global code properties with a parallel symmetric turbo-code $(07, 05)_8$, for the different VLCs of Tab. 1.

	VLC	$\mathrm{RVLC}_{d_f=1}$	$\mathrm{RVLC}_{d_f=2}$
Free distance	7	7	7
Interleaver gain with long interleavers	1	1	2
Interleaver gain at high SNR	2	2	4
Int. gain against desynch., with long interleavers	1	2	2

The bit-clock (resp. symbol-clock) transformation is more accurate if N_b (resp. N_s) is fixed. Still, for long interleavers, the two transformations rapidly tend to the same spectra.

3.4. Concatenation with linear codes

Let GC denote the global code of Fig. 1, and $\otimes$ the operator of serial concatenation of spectra for linear codes (see eq. (7) in [4]). This operator still holds in the case of block-VLCs if, again, we consider the mean spectra. Then the global code spectra are given by the generic relation: $A^{\mathrm{GC}}_{.,.} = A^{\text{b-VLC}}_{.,.} \otimes A^{\mathrm{ECC}}_{.,.}$.

With these spectra and for an ML-decoding of the global code of Fig. 1, the union bounds on the BER and FER are computed with (1), and on the SER with:

$$\mathrm{SER} \leq P_s(e) \triangleq \sum_{s,h} \frac{s}{N_s} A^{\mathrm{GC}}_{s,h}\, P_h.$$

3.5. Simulations and tightness of the bounds

Consider the VLCs of Tab. 1, concatenated with a rate $2/3$ punctured recursive systematic CC (RSCC) of generators $(07, 05)_8$, and a uniform interleaver size is $N_b = 384$. The bound on the $\mathrm{SER_L}$ is compared against the simulation results in Fig. 2. As expected, the proposed bound is tight at medium to high SNRs.

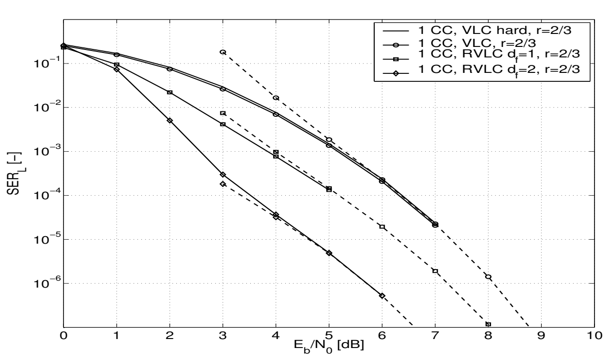

Fig. 2. Simulations (continuous) and union bound (dashed) on the $\mathrm{SER_L}$ for different VLCs concatenated with a punctured CC.

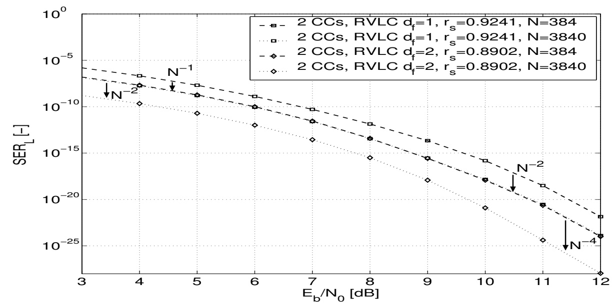

Fig. 3. Illustration of the gains given in Tab. 2, uniform interleavers of length $N_b = 384$ and $N_b = 3840$.

4. INTERLEAVER GAINS

It can be shown that the interleaver gains demonstrated in [4] can be extended to our scheme. Only the final results are reported.

Consider the scheme of Fig. 1 with a parallel symmetric turbo-code as ECC, with RSCCs as constituent codes. The interleavers gains for BER and SER are then given asymptotically by:

- for large E_b/N_0: $\quad 1 - 2w_{mm} + \lfloor w_{mm}/d_f^{\mathrm{VLC}} \rfloor$,
- for large N_b if $d_f^{\mathrm{VLC}} \geq 2$: $\quad -2\lfloor (d_f^{\mathrm{VLC}} + 1)/2 \rfloor$,
- for large N_b if $d_f^{\mathrm{VLC}} = 1$: $\quad -1$,

where w_{mm} is the minimum w such that $A^{\mathrm{GC}}_{w,h=d_f^{\mathrm{GC}}} \neq 0$. Compared to a turbo-code alone, the gains can be doubled if $d_f^{\mathrm{VLC}} \geq 2$, as for $\mathrm{RVLC}_{d_f=2}$ (reversible VLC) in Tab. 2.

To illustrate this, the bound on the $\mathrm{SER_L}$ is plotted in Fig. 3 for two interleaver sizes $N_b = 384$ and $N_b = 3840$. Note that the gains computed in Tab. 2 are well confirmed in Fig. 3.

In Tab. 2, we added an unusual extra gain which is the interleaver gain against desynchronization errors (the most disastrous ones). This one is greater for all RVLCs even if $d_f = 1$, hence their better performance. This generalizes the results of [2]. In addition, the proposed scheme offers an interleaver gain even for the FER. This explains why we obtained in [8] great coding gains for the image error rate with RVLCs and turbo-codes.

5. TURBO-CONVERGENCE OF THE SCHEME

At low SNRs, where the union bound usually underestimates the performance, an efficient analysis tool for turbo-codes is the 2D EXIT chart proposed in [12]. This chart can be extended to our scheme (VLC with a turbo-code) by adding a third dimension, brought by the VLC SISO module, as illustrated in Fig. 5. The gray surfaces represent the information produced by the CCs of the turbo-code. Note that the intersection between these gray surfaces and the plane $VLC_{\mathrm{out}} = 0$ give the well known 2D EXIT chart for the turbo-code alone, which shows in this case that without the VLC SISO module, the decoding trajectory would get stuck at about $CC1_{\mathrm{out}} = CC2_{\mathrm{out}} = 0.2$. But if we add the VLC SISO module in the iterative decoding (the white surface in the figure) a 3D tunnel is created through which the decoding trajectory (the thick white line) can pass in order to reach an extrinsic information of 1. The minimum SNR required for the 3D tunnel to exist is called the *convergence threshold*. It is the threshold above which a turbo-decoder with sufficiently long interleavers can converge.

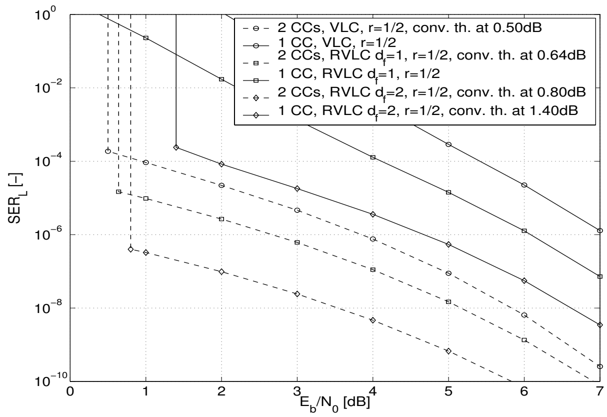

Fig. 4. Union bounds on the $\mathrm{SER_L}$ and convergence thresholds predicted by the 3D EXIT chart for different schemes

A nice property of these 3D EXIT charts is the simplicity of their construction. Indeed, with a few realistic assumptions, any of the 3D surfaces can be constructed from its corresponding 2D curve. Briefly, for three SISO modules a, b, c, the transformation from 2D to 3D for the module c is given by:

$$I_{c,\text{out}}^{3\text{D}}(I_a, I_b) = I_{c,\text{out}}^{2D}\Big(J\big(\sqrt{[J^{-1}(I_a)]^2 + [J^{-1}(I_b)]^2}\big)\Big)$$

where I_a is the information produced by the module a, I_b by the module b; $I_{c,\text{out}}^{3\text{D}}(.,.)$ is the 3D surface, $I_{c,\text{out}}^{2\text{D}}(.)$ the 2D curve as it can be found in [9]; $J(.)$ is defined in [12].

In Fig. 4, the proposed bounds and the convergence thresholds for different VLCs are given. The source is the English alphabet, see [11]. The ECC (one or two CCs) is punctured such that the overall code rate is constant. The figure shows that one CC is not sufficient to achieve good performance/convergence whatever VLC is used. It shows also that, depending on the application, $\text{RVLCs}_{d_f=1}$ with two CCs achieve good performance (and moreover at a lower convergence threshold as compared to $\text{RVLCs}_{d_f=2}$), while with only one CC, such $\text{RVLCs}_{d_f=1}$ behave very badly. This is a new result which differs from the conclusions drawn in [2] where only one CC was envisaged.

Besides convergence analysis, the 3D EXIT chart allows also to optimize the schedule of the iterations between the three SISO modules. Optimally, the VLC SISO module is only used at the beginning to pass through the 3D tunnel and to accelerate the convergence, and at the end to correct the residual errors. With such an optimized schedule, the turbo-decoder we proposed in [9] is surprisingly of lower complexity than the one of a turbo-code alone.

6. SUMMARY AND CONCLUSIONS

We proposed a method to compute the different mean distance spectra of variable length codes (VLCs) turbo-concatenated with linear codes, as well as bounds on the error rates at medium to high SNRs. Also, in the case of VLCs protected by turbo-codes, EXIT charts are extended to three dimensions in order to predict the convergence at low SNRs. Moreover, these charts can also be used to optimize the schedule within the decoding iterations.

As a conclusion, since the developed tools are valid for a wide variety of schemes, this work will help analyze and design other joint source-channel turbo-schemes explored in the literature.

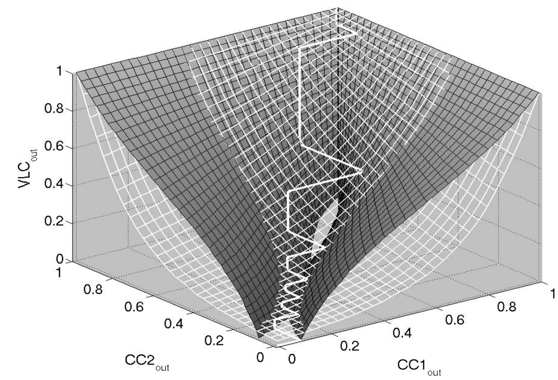

Fig. 5. 3D EXIT chart and decoding trajectory. Each dimension corresponds to the information produced by one SISO module.

7. REFERENCES

[1] R. Bauer and J. Hagenauer, "Iterative Source/Channel-Decoding Using Reversible Variable Length Codes", *DCC Conference*, pp. 93-102, Snowbird, Mar. 2000.

[2] R. Bauer and J. Hagenauer, "On variable length codes for iterative source/channel decoding", *DCC Conference*, pp. 273-282, Snowbird, Mar. 2001.

[3] S. Benedetto, G. Montorsi, "Unveiling turbo codes: some results on parallel concatenated coding schemes", *IEEE Trans. Inf. Th.*, vol. 42, pp. 409-428, Mar. 1996.

[4] S. Benedetto, D. Divsalar, G. Montorsi and F. Pollara "Serial Concatenation of Interleaved Codes: Performance Analysis, Design and Iterative Decoding", *IEEE Trans Inf. Th.*, vol. 44, pp. 909-926, May 1998.

[5] V. Buttigieg, "Variable-Length Error-Correcting Codes", *Ph.D. thesis*, Dept. of Elect. Engineering, University of Manchester, England, 1995.

[6] L. Guivarch, J. C. Carlach and P. Siohan, "Joint source-channel soft-decoding of Huffman codes with turbo-codes", *DCC conference*, Snowbird, pp. 83-92, Mar. 2000.

[7] A. Guyader, E. Fabre, C. Guillemot, and M. Robert, "Joint Source-Channel Turbo Decoding of Entropy-Coded Sources", *IEEE J.S.A. Commun.*, vol. 19, pp. 1680-1696, Sept. 2001.

[8] X. Jaspar, L. Vandendorpe, "Three SISO Modules Joint Source-Channel Turbo-Decoding of Variable Length Coded Images", *SCC conference*, Erlangen, pp. 279-286, Jan. 2004.

[9] X. Jaspar, L. Vandendorpe, "New Iterative Decoding of Variable Length Codes with Turbo Codes," ICC conference, pp. 2606-2610, Paris, June 2004.

[10] J. Kliewer and R. Thobaben, "Parallel concatenated joint source-channel coding", *IEE Elect. Letters*, Vol. 39, pp. 1664-1666, Nov. 2003.

[11] K. Lakovic and J. Villasenor, "Combining Variable Length Codes and Turbo Codes", *VTC Conference*, vol. 4, pp. 1719-1723, Birmingham, May 2002.

[12] S. ten Brink, "Convergence Behavior of Iteratively Decoded Parallel Concatenated Codes", *IEEE Trans. Comm.*, vol. 49, pp. 1727-1737, Oct. 2001.