LINEAR EQUALIZERS FOR FLAT RAYLEIGH MIMO CHANNELS

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ABSTRACT

We consider linear detectors for MIMO systems, i.e., multi-antenna systems where linear equalizers are employed to remove spatial interference. We analyze the behavior of linear equalizers through outage probability. The MMSE equalizer was found to behave in unexpected ways. Contrary to the usual intuition, the performance of MMSE and zero-forcing equalizers may not coincide at high-SNR. This is especially true at low spectral efficiencies, where MMSE equalizer may achieve full spatial diversity.

1. INTRODUCTION

Multiple-input multiple output (MIMO) systems experience interference between signals transmitted simultaneously from transmit antennas. Various detection methods can be used to remove the spatial interference.

Performance of nulling and cancelling detectors have been studied previously (see [1, 2, 3] among others). Linear equalizers are less complex and it is known that their performance is inferior to nulling and cancelling methods. However, [4] reports that some linear equalizers may perform surprisingly different.

This paper studies linear detection methods in flat fading MIMO channels. We evaluate their outage probability and derive their diversity order. We show that in high spectral efficiencies minimum mean-square linear equalizers (MMSE-LE) perform almost the same as zero-forcing linear equalizers (ZF-LE). However, in low spectral efficiencies MMSE-LE performs better than ZF-LE in entire SNR range, and may achieve full spatial diversity.

2. LINEAR EQUALIZERS

The input-output system model for flat fading MIMO channel with M transmit and $N \geq M$ receive antennas is

$$\mathbf{r} = \mathbf{H}\mathbf{c} + \mathbf{n} \ , \tag{1}$$

where $\mathbf{c}$ is the $M \times 1$ transmitted vector, $\mathbf{n} \in \mathcal{C}^{N\times 1}$ is the noise vector, and $\mathbf{r}$ is the $N \times 1$ received vector at a given time instant.

The ZF equalizer is $\mathbf{F}_{\text{ZF}} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$, which transforms the received signal to

$$\hat{\mathbf{r}} = \mathbf{F}_{\text{ZF}}\,\mathbf{r} = \mathbf{c} + (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{n} \ . \tag{2}$$

The MMSE equalizer is $\mathbf{F}_{\text{MMSE}} = (\mathbf{H}^H\mathbf{H} + \rho^{-1}\mathbf{I})^{-1}\mathbf{H}^H$, where ρ is the receive SNR.

Since the symbols are detected individually, the SINR of the individual symbols determines the performance. The detection noise in (2), $\tilde{\mathbf{n}} \stackrel{\triangle}{=} (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{n}$, is a complex Gaussian vector with zero-mean and covariance matrix

$$\mathbf{R}_{\tilde{\mathbf{n}}} = \mathcal{E}\left((\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{n}\right) = \sigma_n^2(\mathbf{H}^H\mathbf{H})^{-1} \ . \tag{3}$$

The associated SINR is $\gamma_k = \mathcal{E}_x/\mathbf{R}_{\tilde{\mathbf{n}}}(k,k)$, which can be shown to be a chi-square random variable with $2(N-M+1)$ degrees of freedom [5].

The SINR of the kth symbol of MMSE detector is determined by noise and residual interference, and is given by [4]

$$\begin{aligned}\gamma_k &= \mathbf{h}_k^H\left(\hat{\mathbf{H}}_k\hat{\mathbf{H}}_k^H + \rho^{-1}\mathbf{I}\right)^{-1}\mathbf{h}_k & (4)\\ &= \frac{1}{(\mathbf{I}+\rho\mathbf{H}^H\mathbf{H})_k^{-1}} - 1 \ , & (5)\end{aligned}$$

where $\mathbf{h}_k$ is the kth column of the channel matrix $\mathbf{H}$ and removing this column from $\mathbf{H}$ gives $\hat{\mathbf{H}}_k \in \mathcal{C}^{N\times(M-1)}$.

Equation (4) shows that γ_k is a quadratic form whose statistics has been derived in [6] as follows. Considering the random matrix $\hat{\mathbf{H}} \in \mathcal{C}^{N\times(M-1)}$ and the random vector $\mathbf{h} \in \mathcal{C}^N$, the quadratic form $Y = \mathbf{h}^H(\hat{\mathbf{H}}\hat{\mathbf{H}}^H + \rho^{-1}\mathbf{I})^{-1}\mathbf{h}$ has the CDF

$$F_Y(y) = 1 - \exp(-\frac{y}{\rho})\sum_{n=1}^{N}\frac{A_n(y)}{(n-1)!}\left(\frac{y}{\rho}\right)^{n-1} \tag{6}$$

where the auxiliary functions $A_n(y)$ are given by

$$A_n(y) = \begin{cases} 1 & N \geq M+n-1 \\ \frac{1+\sum_{i=1}^{N-n} C_i y^i}{(1+y)^{M-1}} & N < M+n-1\,. \end{cases} \ , \tag{7}$$

and C_i is the coefficient of y^i in $(1+y)^{M-1}$ [6].

In general, the SINR of the output symbols of the MMSE receiver are correlated, unlike those of the zero-forcing equalizer.

3. OUTAGE PROBABILITY IN SEPARATE SPATIAL ENCODING

Separate spatial encoding is the method where the data stream is demultiplexed to several sub-streams, each for one transmit antenna (e.g. V-BLAST). The outage event $\mathcal{O}$ occurs when any of the sub-channels cannot support the rate that is assigned to it. In our analysis, we consider equal rate for the sub-channels, however, it is also possible to have a non-uniform rate assignment.

After linear transformation, the mutual information between the elements of $\hat{\mathbf{r}}$ and the transmitted data vector $\mathbf{c}$ is $\mathcal{I}(c_k;\hat{r}_k) = \log(1+\gamma_k)$. Assuming target rate R, the outage probability $\Pr(\mathcal{O})$ is :

$$\begin{aligned}\Pr(\mathcal{O}) &= 1-\Pr\left(\bigcap_{k=1}^{M}\left\{\mathcal{I}(c_k;\hat{r}_k)\geq\frac{R}{M}\right\}\right) . \\ &= 1-\left(\Pr\left(\mathcal{I}(c_k;\hat{r}_k)\geq\frac{R}{M}\right)\right)^M \\ &\approx M\Pr\left(\mathcal{I}(c_k;\hat{r}_k)<\frac{R}{M}\right) . \end{aligned} \quad (8)$$

Notice that in the above derivation, we have used the fact that sub-channel outage events are independent, which is a valid assumption for ZF-LE. While this assumption does not hold for MMSE-LE, it does give a useful approximation that provides the diversity order.

Using the CDF of $\chi_{2(N-M+1)}$ in the evaluation of (8) gives the outage probability for ZF-LE which is

$$\begin{aligned}\Pr(\mathcal{O}) &\approx MF_Y\left(\frac{2^{R/M}-1}{\rho}\right) \\ &\doteq \frac{M(2^{R/M}-1)^{L+1}}{(L+1)!}\rho^{-(L+1)} , \end{aligned} \quad (9)$$

where $\doteq$ denotes asymptotic equivalence, and $L=N-M$. Equation (9) shows the diversity order $L+1$ for the ZF-LE.

Using (6) in the evaluation of (8) results in the outage probability of MMSE-LE

$$\begin{aligned}\Pr(\mathcal{O}) &\approx MF_Y\left(2^{\frac{R}{M}}-1\right) \\ &\doteq \frac{y^{L+1}}{(L+1)!}\cdot\frac{y^{M-1}}{(1+y)^{M-1}}\rho^{-(L+1)}\Big|_{y=2^{\frac{R}{M}}-1} \end{aligned} \quad (10)$$

which also shows the diversity order $L+1$ for the MMSE-LE, the same as that of ZF-LE. However, the two outage probabilities are not exactly the same. The ratio of the outage probability of (9) to (10) is:

$$\frac{\Pr(\mathcal{O})_{\text{ZF}}}{\Pr(\mathcal{O})_{\text{MMSE}}} = \frac{(1+y)^{M-1}}{y^{M-1}}\Big|_{y=2^{\frac{R}{M}}-1} = \left(\frac{2^{\frac{R}{M}}}{2^{\frac{R}{M}}-1}\right)^{M-1} . \quad (11)$$

Note that the ratio of outage probabilities in (11) remains fixed regardless of SNR and it only depends on the relative target rate $\frac{R}{M}$. When $\frac{R}{M}$ is small the outage probability of ZF-LE becomes larger than that of MMSE-LE. The ratio (11) approaches one when $\frac{R}{M}$ is large (see Section 5).

Generalization of the above results to non-uniform rate assignment is straightforward. Uniform and non-uniform rate assignment have the same diversity, though they have different performance.

It is also possible to obtain the diversity-multiplexing tradeoff, introduced in [7], for ZF-LE and MMSE-LE. Substituting $R = r\log\rho$, a similar derivation for ZF-LE as in (9) leads to

$$\begin{aligned}\Pr(\mathcal{O}) &\approx MF_Y\left(\frac{\rho^{\frac{r}{M}}-1}{\rho}\right) \\ &\doteq \frac{M}{(L+1)!}\rho^{-(L+1)(1-\frac{r}{M})^+} , \end{aligned}$$

where $(x)^+ = \max(0,x)$. The above result indicates that ZF-LE achieves the *diversity gain* of $d(r) = (L+1)(1-\frac{r}{M})$. Similarly for MMSE-LE, the derivation which led to (10) gives:

$$\begin{aligned}\Pr(\mathcal{O}) &\approx MF_Y\left(\rho^{\frac{r}{M}}-1\right) \\ &\doteq \frac{M}{(L+1)!}\rho^{-(L+1)(1-\frac{r}{M})^+} . \end{aligned}$$

4. OUTAGE PROBABILITY IN JOINT SPATIAL ENCODING

Joint spatial encoding is the method where the data stream is encoded and then demultiplexed into sub-streams to be sent from the antennas (e.g. D-BLAST). Effectively, each data symbol can contribute to signals of all the transmit antennas. Considering the action of linear equalizers, outage occurs when the aggregate mutual information of all the sub-channels fails to support the target rate.

Assuming the target rate is R, the probability of the outage event $\mathcal{O}$ is

$$\Pr(\mathcal{O}) = \Pr\left(\sum_{k=1}^{M}\mathcal{I}(c_k;\hat{r}_k)<R\right) \quad (12)$$

$$= \Pr\left(\prod_{k=1}^{M}(1+\gamma_k)<2^R\right) . \quad (13)$$

The SINR of the sub-channels of ZF-LE are independent chi-square random variables with degrees $2(N-M+1)$. Let $Y_k \sim \chi_{2(N-M+1)}$, $k=1,\cdots,M$. The outage probability of ZF-LE is given by the CDF of the random variable

$$\prod_{k=1}^{M}(1+Y_k) = 1+\sum_{k=1}^{M}Y_k+\cdots+\prod_{k=1}^{M}Y_k . \quad (14)$$

Among the components of the above random variable, the last term, which is the product of Y_k's, determines the diversity order. Through recursion, one can show[1] that $Y_1 \cdot Y_2 \cdots Y_M$ has diversity order $L+1$. Therefore, the diversity order of ZF-LE is $L+1$. Recalling the results from Section 3, ZF-LE has the same diversity in joint and separate spatial encoding architecture.

To obtain the outage probability of MMSE-LE, we substitute the SINR of MMSE-LE from (5) in (13) which gives:

$$\Pr(\mathcal{O}) = \Pr\left(\prod_{k=1}^{M}\left(\mathbf{I}+\rho\mathbf{H}^H\mathbf{H}\right)_k^{-1} > 2^{-R}\right) . \quad (15)$$

The involvement of the diagonal elements of the random matrix $\left(\mathbf{I}+\rho\mathbf{H}^H\mathbf{H}\right)^{-1}$ makes further analysis intractable. Therefore, we proceed to provide an upper bound to this probability. The sum mutual information in (12) is

$$\begin{aligned}
-\sum_{k=1}^{M}\mathcal{I}(c_k;\hat{r}_k) &= \sum_{k=1}^{M}\log\left(\left(\mathbf{I}+\rho\mathbf{H}^H\mathbf{H}\right)_k^{-1}\right) \\
&\geq M\log\left(\sum_{k=1}^{M}\frac{1}{M}\left(\mathbf{I}+\rho\mathbf{H}^H\mathbf{H}\right)_k^{-1}\right) \\
&= M\log\left(\frac{1}{M}\sum_{k=1}^{M}\frac{1}{1+\rho\lambda_k}\right) , \quad (16)
\end{aligned}$$

where the inequality is due to Jensen's inequality, and λ_k's are the eigenvalues of the Wishart matrix $\mathbf{H}^H\mathbf{H}$. Substituting (16) into (12) gives:

$$\Pr(\mathcal{O}) \leq \Pr\left(\sum_{k=1}^{M}\frac{1}{1+\rho\lambda_k} \geq M2^{-\frac{R}{M}}\right) . \quad (17)$$

To evaluate the above probability, we need the joint PDF of the eigenvalues of $\mathbf{H}^H\mathbf{H}$. Assuming $N \geq M$, the joint PDF of the ordered eigenvalues λ_k's, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$, is

$$f_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) = K_{M,N}\prod_{i=1}^{M}\lambda_i^{N-M}\prod_{i<j}(\lambda_i-\lambda_j)^2\exp\left(-\sum_i\lambda_i\right) ,$$

where $K_{M,N}$ is a normalizing constant [7].

The evaluation of (17) for a specific outage rate R is rather difficult, due to the shape of the outage region. However, one can calculate the bound for small and large values of R where the the outage region can be approximated by regions with simpler shapes.

For a MIMO channel with $M = 2$ and $N \geq 2$, the bound (17) on outage probability becomes

$$\Pr(\mathcal{O}) \leq \Pr\left(\frac{1}{1+\rho\lambda_1}+\frac{1}{1+\rho\lambda_2} \geq 2^{1-\frac{R}{2}}\right) \quad (19)$$

For small values of R, the outage region is an isosceles right triangle with the side $\lambda_1+\lambda_2 = c_2$, where $c_2 \triangleq \frac{2-b}{\rho(b-1)}$ and $b \triangleq 2^{1-\frac{R}{2}}$. When $N \geq 2$ the outage probability bound (17), up to the scaling factor $K_{2,N}$, is

$$\begin{aligned}
\Pr(\mathcal{O}) &\leq \int_0^{c_2} e^{-\lambda_1}\lambda_1^{N-2}\int_0^{c_2-\lambda_1}\lambda_2^{N-2}(\lambda_1-\lambda_2)^2e^{-\lambda_2}d\lambda_2\,d\lambda_1 \\
&\doteq \frac{2(N-1)!(N-2)!}{(2N)!}\left(\frac{2-b}{b-1}\right)^{2N}\rho^{-2N} , \quad (20)
\end{aligned}$$

where the achieved diversity is $2N$. This is surprising because $2N$ is the maximum achievable diversity order.

For large values of R, the outage region is approximated by two orthogonal strips. The strips are defined as $0 \leq \lambda_2$, $0 \leq \lambda_1 \leq \hat{c}_2$ and $0 \leq \lambda_1$, $0 \leq \lambda_2 \leq \hat{c}_2$, where $\hat{c}_2 \triangleq \frac{1-b}{b\rho}$. The outage probability bound, up to the scaling factor $2K_{2,N}$, is

$$\begin{aligned}
\Pr(\mathcal{O}) &\leq \int_0^{\hat{c}_2} e^{-\lambda_1}\lambda_1^{N-2}\int_0^{\infty}\lambda_2^{N-2}(\lambda_1-\lambda_2)^2e^{-\lambda_2}d\lambda_2\,d\lambda_1 \\
&\doteq N(N-2)!\left(\frac{1-b}{b}\right)^{N-1}\rho^{-(N-1)} , \quad (21)
\end{aligned}$$

which indicates that the upper bound[2] has the diversity $N-1 = L+1$, where $L = N-M$. In the calculation of (21), the intersection of the two strips is calculated twice. This portion of integral, which decays as fast as $\rho^{-2(N-1)}$, does not affect the asymptotic behavior of (21).

The previous results of the case $M = 2, N \geq 2$ can be extended to arbitrary values of M and $N \geq M$, by generalizing the outage regions in low and high spectral efficiencies.

5. SIMULATION RESULTS

We consider a MIMO system with $M = N = 2$. The outage probability of the linear equalizers in the separate architecture is shown in Figure 1. As expected, both linear detectors show diversity order of one, regardless of the target rate. For higher values of R they perform almost the same. But, for lower values of R, MMSE-LE performs better than ZF-LE throughout the SNR region. The dependency of the relative performance of these equalizers on the target rate R is in agreement with (11).

In Figure 2, the outage probability of the unconstrained receiver and linear equalizers in a joint spatial encoding architecture are shown. The ZF-LE equalizer has diversity one regardless of R, as expected from Section 4. Surprisingly, MMSE-LE shows diversity rate that depends on R: for lower values of R the diversity order is very close to

[1]Due to space limitation the derivation is not presented.

[2]In Section 5, we show that the upper bound is tight in high spectral efficiency region.

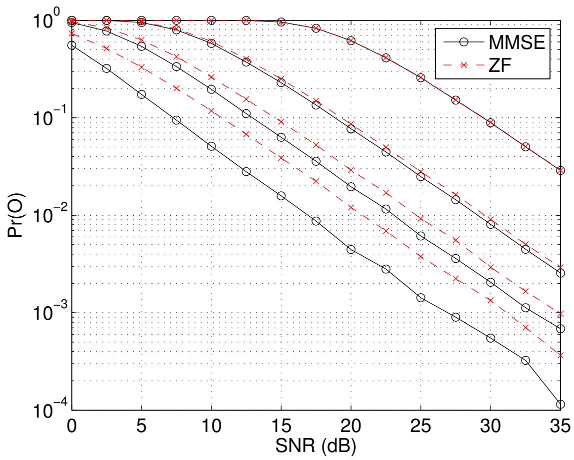

Fig. 1. Separate spatial encoding. From left to right R=1,2,4,10 bits/sec/Hz.

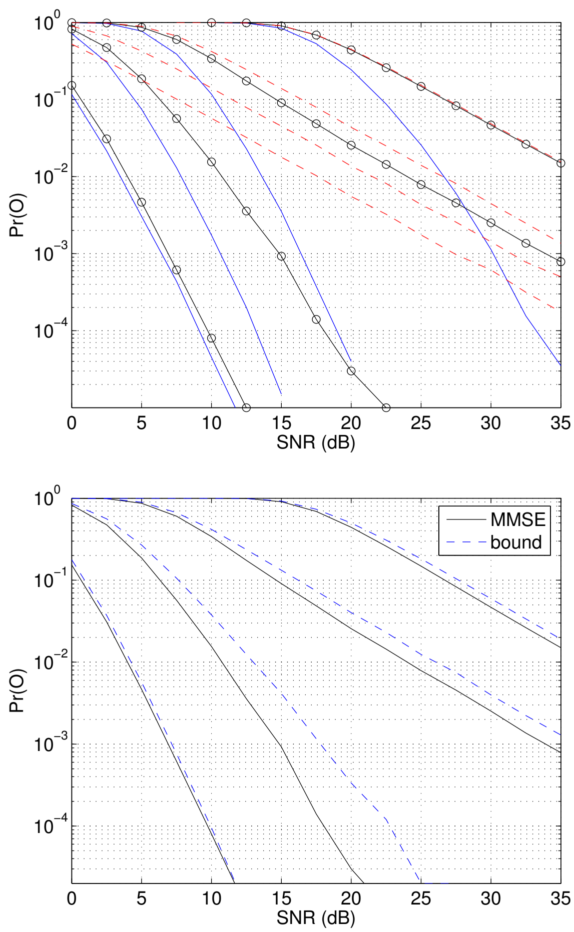

Fig. 2. Joint spatial encoding. Top: Unconstrained receiver (solid line), MMSE-LE (solid line with ∘) and ZF-LE (dashed line). Bottom: MMSE-LE and the upper bound (17). From left to right R=1,2,4,10 bits/sec/Hz.

that of the unconstrained receiver, and for higher values of R its diversity becomes the same as the diversity order of ZF-LE. These results are in agreement with the analysis in Section 4. Figure 2 also shows the tightness of the upper bound (17) for small and large R. Though the bound is loose for the intermediate values of R, it does predict diversity order varying with R.

6. CONCLUSION

We present new results on the performance of linear equalizers in MIMO channels, and calculate their diversity order. Our analytical and experimental results show that MMSE linear equalizers have outage probability that may decay as fast as the outage probability of the unconstrained receiver, or as slowly as that of ZF linear equalizers, depending on spectral efficiency,

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