SPATIAL LIMITS TO MUTUAL INFORMATION SCALING IN MULTI-ANTENNA SYSTEMS

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ABSTRACT

Previous results have shown significant capacity gains by employing multiple antennas at both transmitter and receiver, however, due to physical size restraints (particularly at the receiver) these may not be obtained. In this paper we consider the capacity behaviour of multi-antenna systems when the receiver sampling is constrained to a finite region of space. By characterizing the wavefield generated at the receiver due to transmitted signals and the scattering environment, a theoretically derived sampling threshold is shown to exist, at which the capacity growth is reduced from linear to logarithmic with increasing number of sampling outputs. Furthermore, this threshold is shown to be linearly dependent on the receiver region radius within which the sampling is constrained, and is independent of the sampling characteristics such as the antenna properties, array geometry, and/or array signal processing.

1. INTRODUCTION

Multiple-Input Multiple-Output (MIMO) communications systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [1] showed the potential for significant capacity increases in wireless channels via spatial multiplexing with sparse arrays. However, in reality the capacity is significantly reduced when the antennas are constrained to within finite regions as the signals received by different antennas become correlated [2]. Previous studies have given insights into asymptotic capacity (w.r.t. the number of antennas) of fixed length linear arrays [3,4], and arrays within fixed volumes [5], however, of significant interest is the limits to capacity for finite numbers of antennas in spatially constrained arrays.

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In [6] it was shown that there is a saturation point for a spatially constrained circular array of fixed radius, at which there is no further increase in capacity with increasing number of antennas. In this paper we extend this result to arbitrary sampling of a fixed radius aperture and demonstrate a saturation point which is independent of the method of sampling (directional antennas, coupled or uncoupled antenna arrays, multimode antennas, etc).

We approach the MIMO capacity from a physical wavefield perspective. By using the underlying physics of free-space wave propagation we explore fundamental spatial sampling limits imposed by the basic laws governing wavefield behavior. In particular, using a modal expansion for free-space wave propagation we show that there exists a threshold in sampling, which depends on the radius of the aperture being sampled, at which the capacity scaling is reduced from linear to logarithmic. This result is shown to be independent of the sampling method and provides a threshold for future MIMO system designs.

2. WAVEFIELD SAMPLING

In a narrowband MIMO system with no polarization diversity the signals arriving at the receiver generate a multipath field (scalar wavefield), which is sampled in space using an antenna array. Consider a 2D receiver where the incoming signals from direction ψ are given by the complex valued density $g(\psi) \in L^2(\mathbb{S}^1)$. Let $\boldsymbol{r} \in \mathbb{R}^2$ with $\boldsymbol{r} = (r, \theta)$ then a Hilbert space $\mathcal{H}$ of wavefields $f(\boldsymbol{r})$ satisfying the source-free Helmholtz equation

$$\nabla^2 f(\boldsymbol{r}) + k^2 f(\boldsymbol{r}) = 0, \tag{1}$$

where $k = 2\pi/\lambda$ is the wavenumber, can be written as

$$f(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \langle f, \phi_n \rangle_{\mathcal{H}} \phi_n, \tag{2}$$

where $\{\phi_n \equiv \phi_n(\boldsymbol{r})\}$ is a complete and orthogonal basis set in $\mathcal{H}$, with the natural inner product

$$\langle f_1, f_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} f_1(\boldsymbol{r}) \overline{f_2(\boldsymbol{r})} d\boldsymbol{r}. \quad (3)$$

Consider a device which *samples* the wavefield f and returns Q outputs $\boldsymbol{y} = [y_1, y_2, \cdots, y_Q]'$, given by

$$\boldsymbol{y} = \mathcal{A} f, \quad (4)$$

where $\mathcal{A}$ is a sampling operator, which describes the sampling of the wavefield $f(\boldsymbol{r})$. Let $\alpha_n = \langle f, \phi_n \rangle_{\mathcal{H}}$ and assuming the sampling operator is linear, the output can be expressed as

$$\boldsymbol{y} = \mathcal{A} f \quad (5a)$$

$$= \sum_n \alpha_n \mathcal{A} \phi_n \quad (5b)$$

$$= \boldsymbol{A} \boldsymbol{\alpha} \quad (5c)$$

where $\boldsymbol{A}$ is defined as the sampling matrix with columns given by vectors $\boldsymbol{a}_n \triangleq \mathcal{A}\phi_n$, which represents the output of the sampler due to the field $\phi_n(\boldsymbol{r})$. The vector $\boldsymbol{\alpha}$ corresponds to the coefficients of the wavefield expansion (2).

Note that the outputs of the sampler are now separated into the product of the independent properties of the spatial sampling matrix, $\boldsymbol{A}$, and that of the coefficients of the wavefield, α_n. It is important to note that the sampling matrix is general and independent of the field, and can describe the sampling via a number of means. For example, an array of Q uncoupled omnidirectional antennas located at positions $\boldsymbol{r}_q$ gives $A_{pq} = \phi_p(\boldsymbol{r}_q)$. Other examples include multi-mode antennas which allow for the excitation of several modes of the same frequency on a single antenna, offering the diversity characteristics similar to those of an antenna array [7, 8]. In this paper we focus on the outputs of such devices rather than the properties of each implementation, this leads to results which are independent of sampling characteristics such as the antenna properties, array geometry, and/or array signal processing.

3. CONVERGENCE OF ERGODIC CAPACITY

Consider a narrowband MIMO system with transmit and receive devices consisting of S inputs and Q outputs, let the transmitted signals be statistically independent equal power components each with a Gaussian distribution, then the ergodic mutual information of the system is given by [1],

$$\widetilde{I} = E\left\{\log_2 \left| \boldsymbol{I}_Q + \frac{\eta}{S} \boldsymbol{H}\boldsymbol{H}^\dagger \right| \right\} \text{ bits/s/Hz}, \quad (6)$$

where $\boldsymbol{H}$ is the normalized $Q \times S$ random flat fading channel matrix describing the gains from the S inputs to the Q outputs and assumed known at the receiver, η is the average signal-to-noise ratio (SNR) at each output , $\boldsymbol{I}_Q$ is the $Q \times Q$ identity matrix, $|\cdot|$ is the determinant operator, and $\dagger$ the Hermitian operator.

Assuming a large number of uncorrelated inputs the ergodic mutual information converges to the deterministic quantity I [6, 9],

$$\lim_{S\to\infty} \widetilde{I} = I \triangleq \log |\boldsymbol{I}_Q + \eta \boldsymbol{R}_Q| , \quad (7)$$

where $\boldsymbol{R}_Q$ is the $Q \times Q$ receiver correlation matrix. This mutual information expression will be accurate for many practical wireless scenarios, where the receiver is often size limited, whilst the base station is less restricted in geometrical size and is able to provide a sufficient number of uncorrelated transmit branches [9].

For a given received wavefield sampler $\mathcal{A}$ the receiver correlation matrix can be written as

$$\boldsymbol{R}_Q = E\{\boldsymbol{y}\boldsymbol{y}^\dagger\} \quad (8a)$$

$$= E\left\{\boldsymbol{A}\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger \boldsymbol{A}^\dagger\right\} \quad (8b)$$

$$= \boldsymbol{A} E\left\{\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger\right\} \boldsymbol{A}^\dagger \quad (8c)$$

$$= \boldsymbol{A}\boldsymbol{\Gamma}\boldsymbol{A}^\dagger, \quad (8d)$$

where $\boldsymbol{\Gamma} = E\left\{\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger\right\}$ is the covariance matrix of the wavefield coefficients α_n. The receiver output correlation matrix decomposition (8d) separates the correlation into two distinct parts: the field coefficient correlation matrix $\boldsymbol{\Gamma}$ giving the statistics of the wavefield, and the deterministic sampling matrix $\boldsymbol{A}$ which describes how the wavefield is sampled and is independent of the field.

Assuming we can sample the field such that all Q outputs are uncorrelated then $\boldsymbol{R}_Q = \boldsymbol{I}_Q$, giving

$$I_{\max} = Q \log(1 + \eta), \quad (9)$$

therefore, in the idealistic situation of zero correlation at both transmitter and receiver we see the maximum capacity scaling is linear in the number of receiver outputs. In this case, the system achieves the equivalent of Q independent nonfading subchannels, each with SNR η. This result agrees with the traditional capacity formulation [10] which is widely used to advocate the use of MIMO systems. However, as shown in the follow sections, if the region over which we can sample (the sampling aperture) is constrained to a finite region of space then once the number of outputs reaches a threshold the scaling is reduced to logarithmic growth. This logarithmic growth is due to an effective increase in the average SNR, caused by the assumption of independent noise at each output, and is widely known as a receiver diversity array gain effect.

4. CHANNEL MODEL

Consider the 2D scattering environment shown in Fig. 1, where the receiver is constrained to sampling within a aperture of radius R, i.e., $\boldsymbol{r} \in \mathbb{B}_R^2$. Let $g_s(\psi)$ represent the effective random complex gain of the scatterers for a transmitted signal x_s arriving at the receiver region from direction ψ via any number paths through the scattering environment. The receiver region $\mathbb{B}_R^2$ is contained within a scatterer free region whose radius R_S is assumed large enough such that any scatterers are farfield to all points within $\mathbb{B}_R^2$.

Since the scatterers are assumed farfield, the wavefield within the aperture can be written as a linear combination of planewaves,

$$f(\boldsymbol{r}) = \int_{\mathbb{S}^1} g(\psi) e^{-ikr\cos(\theta-\psi)} d\psi, \tag{10}$$

where $g(\psi) = \sum_s x_s g_s(\psi)$ represents the total signal from direction ψ at the receiver. Consider the 2D Jacobi-Anger modal expansion of plane waves [11],

$$e^{-ikr\cos(\theta-\psi)} = \sum_{n=-\infty}^{\infty} J_n(kr) e^{in(\theta-\pi/2)} e^{-in\psi}, \tag{11}$$

then the wavefield (10) can be written in the general form of (2) with

$$\phi_n(\boldsymbol{r}) = J_n(kr) e^{in(\theta-\pi/2)}, \tag{12}$$

$$\alpha_n = \int_{\mathbb{S}^1} g(\psi) e^{-in\psi} d\psi, \tag{13}$$

with Bessel functions of the first kind $J_n(\cdot)$.

The elements of the covariance matrix $\boldsymbol{\Gamma}$ of the wavefield coefficients can now be expressed as

$$\gamma_{n,n'} = E\left\{\alpha_n, \overline{\alpha_{n'}}\right\} \tag{14a}$$

$$= \iint E\left\{g(\psi)\overline{g(\psi')}\right\} e^{-in\psi} e^{in'\psi'} d\psi d\psi' \tag{14b}$$

$$= \int G(\psi) e^{-i(n-n')\psi} d\psi \tag{14c}$$

assuming a zero-mean uncorrelated scattering environment where $G(\psi) = E\left\{|g(\psi)^2|\right\}$ is the average power density of the scatterers, normalized such that $\int_{\mathbb{S}^1} G(\psi) d\psi = 1$.

5. SATURATION OF CAPACITY SCALING

To isolate the effects of spatially constraining the array from those of the scattering environment we assume the scatterers have uniform power spectral density $G(\psi) = 1/2\pi$, generating an isotropic diffuse field (often referred to as a *rich* scattering environment) corresponding to independent modes $\gamma_{n,n'}$, giving modal correlation matrix $\boldsymbol{\Gamma} = \boldsymbol{I}$. The mutual information (7) is then given by

$$I = \log\left|\boldsymbol{I}_Q + \eta \boldsymbol{A}\boldsymbol{A}^\dagger\right|, \tag{15}$$

which is a function of SNR η and the properties of the sampling matrix $\boldsymbol{A}$ only.

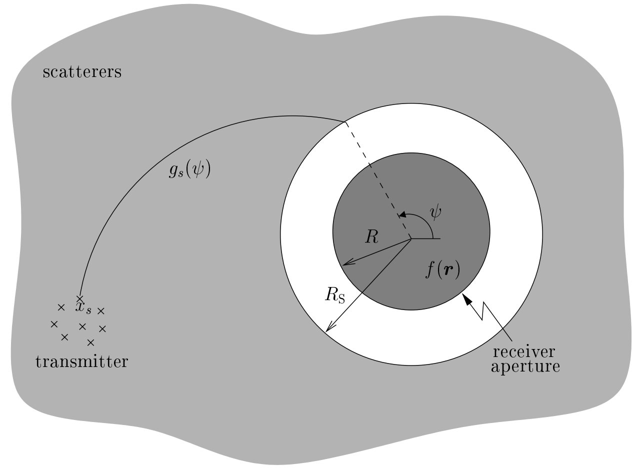

Fig. 1. Two dimensional scattering model for a flat fading MIMO system.

Theorem 1 (sampling saturation point). *For an aperture of radius R define*

$$Q_{sat} \triangleq 2\lceil \pi e R/\lambda \rceil + 1, \tag{16}$$

as the aperture sampling threshold, then for $Q \geq Q_{sat}$ *the mutual information for optimal sampling is well approximated by*

$$I_{sat} \simeq Q_{sat} \log(1 + Q\frac{\eta}{Q_{sat}}). \tag{17}$$

Before giving a proof of Theorem 1 we give the following interpretation:

For a MIMO system within a 2D isotropic diffuse field there exists a saturation point in the number of outputs, which is dependent only on the radius of the sampling aperture, after which further sampling gives only logarithmic mutual information gain.

Proof (sketch). Let μ_m, $m = \{0, 1, \ldots, Q-1\}$ denote the singular values of the aperture sampling matrix $\boldsymbol{A}$, ordered such that $\mu_m \geq \mu_{m+1}$, then we can express the mutual information (7) as

$$I = \sum_{m=0}^{Q-1} \log\left(1 + \eta\mu_m^2\right). \tag{18}$$

Consider two independent sampling matrices A_1 and A_2 operating on an aperture of radius R, giving Q_1 and Q_2 outputs respectively. Denoting $\{\mu_{m,Q_1}\}$ and $\{\mu_{m,Q_2}\}$ as the set of singular values of the aperture sampling matrix for each aperture, and observing that

$$\sum_m \mu_m^2 = \mathrm{trace}(\boldsymbol{A}\boldsymbol{A}^\dagger) = Q, \tag{19}$$

then

$$\frac{1}{Q_1}\sum_{m=0}^{Q_1-1}\mu_{m,Q_1}^2 = \frac{1}{Q_2}\sum_{m=0}^{Q_2-1}\mu_{m,Q_2}^2. \qquad (20)$$

Bessel functions $J_n(z)$, $|n| > 0$ exhibit spatially high pass behavior, that is, for fixed order n, $J_n(z)$ starts small and becomes significant for arguments $z \approx \mathcal{O}(n)$. Therefore, for a fixed argument z, the Bessel function $J_n(z) \approx 0$ for all but a finite set of low order $n \leq N$. In [12] it was shown that $J_n(z) \approx 0$ for $n > \lceil ze/2 \rceil$, with $\lceil\cdot\rceil$ the ceiling operator, therefore, assuming a null response of the sampling device from the zero wavefield, e.g., $\boldsymbol{a}_n = \mathbf{0}, n > N \triangleq \lceil \pi e R/\lambda \rceil$, then

$$\operatorname{rank}(\boldsymbol{A}) = \min\{2N+1, Q\}. \qquad (21)$$

Let $Q_1, Q_2 \geq 2N+1$, and assuming we can optimally sample the aperture such that $\operatorname{rank}(\boldsymbol{A}) = 2N+1$ outputs are uncorrelated, giving constant and equal non-zero singular values, then,

$$\frac{\mu_{m,Q_1}^2}{Q_1} = \frac{\mu_{m,Q_2}^2}{Q_2}, \quad m \in [0, 2N]. \qquad (22)$$

Therefore, letting $Q_1 = Q_{\text{sat}} \triangleq 2N+1$, and $Q_2 = Q \geq Q_{\text{sat}}$, we have $\mu_{m,Q_{\text{sat}}}^2 = 1, \forall m$ and (18) becomes

$$I_{\text{sat}} = Q_{\text{sat}} \log\left(1 + Q\frac{\eta}{Q_{\text{sat}}}\right), \quad Q \geq Q_{\text{sat}}, \qquad (23)$$

which scales logarithmically with Q, hence the maximum capacity growth is reduced from linear to logarithmic once the number of outputs reaches the saturation point $Q_{\text{sat}} = 2N+1$, which scales linearly with the radius of the aperture. □

Fig. 2 shows the mutual information $I_{\max}$ for an increasing number of outputs for the optimal sampler with SNR $10dB$ and sampling aperture radius $R = 0.5\lambda$, giving $Q_{\text{sat}} = 2\lceil \pi e 0.5 \rceil + 1 = 11$. For comparison the capacity of a uniform linear array (ULA) of fixed length $2R$, and a uniform circular array (UCA) of fixed radius R are also shown (e.g. arrays constrained within the fixed region of radius R). As expected from [6] the UCA also saturates around Q_{sat}, however, due the array symmetry of the ULA, it saturates much earlier than optimal sampling due to a much lower sampling matrix rank.

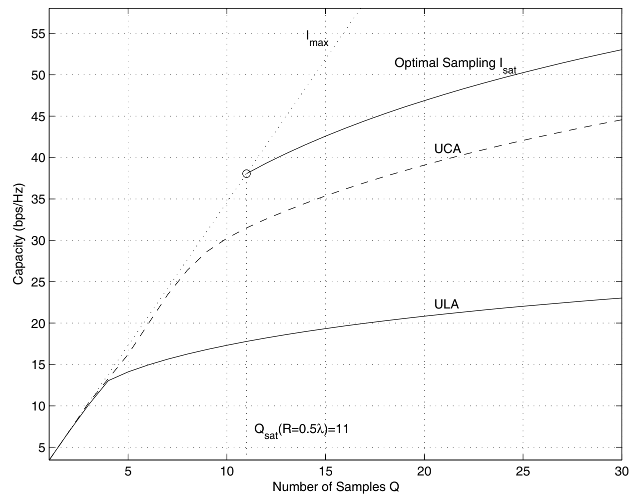

Fig. 2. Capacity

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