MULTIPLE-ANTENNA CAPACITY IN THE LOW-POWER REGIME: CHANNEL KNOWLEDGE AND CORRELATION

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ABSTRACT

The capacity of multiple antenna systems in the low-power regime has gained recently much attention [1]. It turned out that antenna correlation and Ricean factors do not have an impact on the $\frac{E_b}{N_0}_{min}$ but on the slope in bits/s/Hz/(3dB). In this work, we analyze the impact of different types of CSI on the multiple antenna capacity in the low-power regime. We show that the $\frac{E_b}{N_0}_{min}$ is reduced as well as the slope is increased by having channel knowledge at the transmit antenna array. The impact of spatial correlation on the two performance measures is completely characterized. Finally, the theoretical results are illustrated by numerical simulations showing the spectral efficiency over the received $\frac{E_b}{N_0}$ for various systems.

1. INTRODUCTION

Multiple antennas can increase the performance and reliability of transmission over multipath fading channels [2, 3]. The single-user capacity of the multiple antenna channel has been studied extensively with respect to many aspects [4].

In [5], the spectral efficiency in the wideband regime was studied using two novel performance metrics, namely the minimum $\frac{E_b}{N_0}$ and the wideband slope. These quantities characterize the first order behaviour of the capacity at low SNR values. For developing the optimal system design it is necessary to understand the connection between the optimal transmit strategies at low SNR, the minimum $\frac{E_b}{N_0}$, and the wideband slope. Furthermore, it is important to characterize the impact of the channel statistics on the spectral efficiency. In this work, we use the results and expressions from [5] to gain insights into the optimal system design in MIMO channels in the low SNR regime.

For the uncorrelated Rayleigh fading case, these performance metrics were derived in [5, Theorem12] for the informed transmitter case, and [5, Theorem13] for the uninformed transmitter case with perfect CSI at the receiver only. In [1], the uninformed transmitter case with correlation in Rician fading MIMO channels was studied. It turned out, that transmit and receive correlation has no impact on the minimum $\frac{E_b}{N_0}$ but on the wideband slope. This was quantified in [1] by the correlation number.

The contribution of our work is the analysis of the minimum $\frac{E_b}{N_0}$ and the wideband slope $\mathcal{S}_0$ in double correlated Rayleigh fading with different types of CSI at the transmitter, namely perfect and no CSI, and correlation knowledge. We use Majorization theory in order to analyze the impact of correlation. It turns out that

1. for no CSI, the wideband slope is Schur-concave with respect to transmit and receive correlation,
2. for perfect CSI, the minimum $\frac{E_b}{N_0}$ is Schur-concave with respect to transmit and receive correlation and the wideband slope is Schur-convex with respect to correlation,
3. and for covariance knowledge, the minimum $\frac{E_b}{N_0}$ is Schur-concave with respect to transmit correlation and the wideband slope is Schur-concave with respect to receive correlation.

2. SYSTEM MODEL

The transmitter has n_T transmit antennas. The receiver applies n_R receive antennas. The received signal vector $\boldsymbol{y}$ in the quasi-static block flat fading MIMO channel $\boldsymbol{H}$ is given by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

with transmit signal $\boldsymbol{x}$ and additive white Gaussian noise (AWGN) vector $\boldsymbol{n} \sim \mathcal{CN}(0, \sigma_n^2 \boldsymbol{I})$. The channel matrix $\boldsymbol{H}$ for the case in which we have correlated transmit and correlated receive antennas is modeled as

$$\boldsymbol{H} = \boldsymbol{R}_R^{\frac{1}{2}} \cdot \boldsymbol{W} \cdot \boldsymbol{R}_T^{\frac{1}{2}} \tag{2}$$

with transmit correlation matrix $\boldsymbol{R}_T = \boldsymbol{U}_T \boldsymbol{D}_T \boldsymbol{U}_T^H$ and receive correlation matrix $\boldsymbol{R}_R = \boldsymbol{U}_R \boldsymbol{D}_R \boldsymbol{U}_R^H$. $\boldsymbol{U}_T$ and $\boldsymbol{U}_R$ are the matrices with the eigenvectors of $\boldsymbol{R}_T$ and $\boldsymbol{R}_R$ respectively, and $\boldsymbol{D}_T$, $\boldsymbol{D}_R$ are diagonal matrices with the eigenvalues of the matrix $\boldsymbol{R}_T$ and $\boldsymbol{R}_R$, respectively, i.e. $\boldsymbol{D}_T = \text{diag}[\lambda_1^T, ..., \lambda_{n_T}^T]$ and $\boldsymbol{D}_R = \text{diag}[\lambda_1^R, ..., \lambda_{n_R}^R]$. Denote the vector of eigenvalues with $\boldsymbol{\lambda}_T$ and $\boldsymbol{\lambda}_R$ respectively. Without loss of generality, we assume that all eigenvalues are ordered with decreasing order, i.e. $\lambda_1^T \geq \lambda_2^T \geq ... \geq \lambda_{n_T}^T$. The random matrix $\boldsymbol{W}$ has zero-mean independent complex Gaussian identically distributed entries, i.e. $\boldsymbol{W} \sim \mathcal{CN}(0, \boldsymbol{I})$. In the following, we will normalize the transmit and receive correlation matrix and assume that $\sum_{k=1}^{n_T} \lambda_k^T = n_T$ and $\sum_{k=1}^{n_R} \lambda_k^R = n_R$. All logarithms are with respect to base 2 if not otherwise stated.

3. PRELIMINARIES

3.1. Spectral efficiency in the low power regime

In [5], the low-SNR regime has been analyzed and two performance measures namely the $\frac{E_b}{N_0}_{min}$ and the wideband slope S_0 were introduced. The system parameters bandwidth B, transmission rate R, transmit power P and spectral efficiency $C(\frac{E_b}{N_0})$ satisfy the fundamental limit

$$\frac{R}{B} \leq C\left(\frac{E_b}{N_0}\right). \tag{3}$$

At low SNR, the function $C(\frac{E_b}{N_0})$ can be expressed as [5]

$$C\left(\frac{E_b}{N_0}\right) \approx \frac{S_0}{3dB}\left(\frac{E_b}{N_0}\Big|_{dB} - \frac{E_b}{N_0}_{min}\Big|_{dB}\right) \quad (4)$$

with

$$\frac{E_b}{N_0}_{min} = \frac{\log_e 2}{\dot{C}(0)} \quad \text{and} \quad S_0 = \frac{2\left[\dot{C}(0)\right]^2}{-\ddot{C}(0)}. \quad (5)$$

The closer $\frac{E_b}{N_0}$ gets to $\frac{E_b}{N_0}_{min}$ the better is the approximation in (4).

3.2. Measure of spatial correlation

In order to compare two correlation scenarios, the following framework can be applied: For two vectors $\boldsymbol{x}, \boldsymbol{y} \in R^n$ one says that the vector $\boldsymbol{x}$ majorizes the vector $\boldsymbol{y}$ and writes $\boldsymbol{x} \succ \boldsymbol{y}$ if

$$\sum_{k=1}^{m} x_k \geq \sum_{k=1}^{m} y_k \quad \forall m = 1, ..., n-1, \text{ and } \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k.$$

A real-valued function Φ defined on $\mathcal{A} \subset R^n$ is said to be *Schur-convex* on $\mathcal{A}$ if from $\boldsymbol{x} \succ \boldsymbol{y}$ on $\mathcal{A}$ follows $\Phi(\boldsymbol{x}) \geq \Phi(\boldsymbol{y})$. Similarly, Φ is said to be *Schur-concave* on $\mathcal{A}$ if from $\boldsymbol{x} \succ \boldsymbol{y}$ on $\mathcal{A}$ follows $\Phi(\boldsymbol{x}) \leq \Phi(\boldsymbol{y})$.

For further information about majorization theory see [6]. The following definition provides a measure for comparison of two covariance matrices. E.g. the transmit correlation matrix $\boldsymbol{R}_T^1$ is more correlated than $\boldsymbol{R}_T^2$ if and only if $\sum_{l=1}^{m} \lambda_l^{T,1} \geq \sum_{l=1}^{m} \mu_l^{T,2}$ for $m = 1...n-1$ and $\sum_{l=1}^{n} \lambda_1^{T,1} = \sum_{l=1}^{n} \lambda_1^{T,2}$.

It can be shown that vectors with more than two components cannot be totally ordered. This is a problem of all possible orders for comparing correlation vectors. The case in which the transmit antennas are fully correlated corresponds to $\lambda_1^T = n_T$, $\lambda_2^T = ... = \lambda_{n_T}^T = 0$. The case in which the transmit antennas are fully uncorrelated corresponds to $\lambda_1^T = \lambda_2^T = ... = \lambda_{n_T}^T = 1$.

We need the following result (see [6, Theorem 3.A.4]) which is sometimes called Schur's condition. It provides an approach for testing whether some vector valued function is Schur-convex or not. Schur's condition for the Schur-convexity of a symmetric function $f(\boldsymbol{x})$ is given as [6, p. 57]

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0 \quad (6)$$

for all $x_1, x_2 \in \mathcal{I}^n$. Furthermore, $f(\boldsymbol{x})$ is a Schur-concave function on $\mathcal{I}^n$ if $f(\boldsymbol{x})$ is symmetric and

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \leq 0 \quad (7)$$

for all $x_1, x_2 \in \mathcal{I}^n$.

4. MINIMUM $\frac{E_b}{N_0}$ AND WIDEBAND SLOPE $\mathcal{S}_0$ FOR DIFFERENT CSI SCENARIOS

4.1. Uninformed transmitter

In [1, 7], the two performance measures in (5) were computed for the MIMO channel without channel state information (CSI) at the transmitter and with perfect CSI at the receiver. In this case the ergodic capacity as a function of the SNR[1] $\rho = \frac{\mathbb{E}[||\boldsymbol{x}||^2]}{N_0}$ is given by

$$C_{nCSI}(\rho) = \mathbb{E}\left[\log\det\left(\boldsymbol{I} + \frac{\rho}{n_T}\boldsymbol{H}\boldsymbol{H}^H\right)\right]. \quad (8)$$

The first and second derivative of (15) is given by

$$\dot{C}_{nCSI}(0) = \frac{1}{n_T}\mathbb{E}\text{ tr }\left(\boldsymbol{H}\boldsymbol{H}^H\right) = n_R \quad (9)$$

$$\ddot{C}_{nCSI}(0) = -\frac{1}{n_T^2}\mathbb{E}\text{ tr }\left(\left[\boldsymbol{H}\boldsymbol{H}^H\right]^2\right) = n_T^2 \text{ tr } \boldsymbol{R}_R^2 + n_R^2 \text{ tr } \boldsymbol{R}_T^2. \quad (10)$$

As a result, the minimum $\frac{E_b}{N_0}$ and the wideband slope is given by [5, Theorem 13]

$$\frac{E_b}{N_0}_{min}^{nCSI} = \frac{\log_e 2}{n_R} \quad (11)$$

$$S_0^{nCSI} = \frac{2n_T^2 n_R^2}{n_T^2 \sum_{k=1}^{n_R}(\lambda_k^R)^2 + n_R^2 \sum_{k=1}^{n_T}(\lambda_k^T)^2}. \quad (12)$$

Note that we focus on the transmitted $\frac{E_b}{N_0}$ as in [1].

In (11) and (12) the two expressions for the minimum $\frac{E_b}{N_0}$ and the widebandslope are given. Obviously, the transmit and receive correlation has no impact on the minimum $\frac{E_b}{N_0}$ but on the slope S_0. The following theorem characterize the impact of correlation.

Lemma 1 *Fix the receive correlation $\boldsymbol{\lambda}_R$. The wideband slope S_0^{nCSI} as a function of the transmit correlation $S_0(\boldsymbol{\lambda}_R)$ is Schur-concave, i.e.*

$$\boldsymbol{\lambda}_T^1 \succeq \boldsymbol{\lambda}_T^2 \Longrightarrow S_0^{nCSI}(\boldsymbol{\lambda}_T^1) \leq S_0^{nCSI}(\boldsymbol{\lambda}_T^2). \quad (13)$$

For fixed transmit correlation, the wideband slope S_0^{nCSI} is Schur-concave with respect to the receive correlation, i.e.

$$\boldsymbol{\lambda}_R^1 \succeq \boldsymbol{\lambda}_R^2 \Longrightarrow S_0^{nCSI}(\boldsymbol{\lambda}_R^1) \leq S_0^{nCSI}(\boldsymbol{\lambda}_R^2). \quad (14)$$

Proof: Consider the function $f(\boldsymbol{x}) = \sum_{k=1}^{n} x_k^2$. This function is Schur-convex because x^2 is a convex function [6, Proposition 3.C.1]. As a result, the function $g(\boldsymbol{x}) = \frac{c_1}{f(\boldsymbol{x})+c_2}$ with $c_1 > 0$ and $c_2 > 0$ is Schur-concave.

Remark: In [1, 7], the dispersion is introduced as the measure of correlation. It can be easily shown following the same steps as in the proof of Theorem 1 that the dispersion of a $n \times n$ Hermitian matrix $\boldsymbol{A}$ defined as

$$\zeta(\boldsymbol{A}) = \frac{\text{tr } \boldsymbol{A}^2}{n}$$

is Schur-convex with respect to the eigenvalues of $\boldsymbol{A}$.

[1]Note that the capacity as a function of SNR is different from the spectral effi ciency as a function of $\frac{E_b}{N_0}$.

Actually, the Schur-concavity of the average mutual information without CSI at the transmitter and perfect CSI at the receiver has been shown in [8] for all SNR values. Therefore the result in Theorem 1 follows as a special case for the low power regime. This is nevertheless notable since different behaviour of capacity and spectral efficiency were reported in some cases.

In figure 1, the spectral efficiency over $\frac{E_b}{N_0}$ is shown for different MIMO systems with uninformed transmitter and perfectly informed receiver. The impact of the number of receive antennas on the minimum $\frac{E_b}{N_0}$ can be observed. In addition to this, the wideband slope $\mathcal{S}_0$ decreases with increasing transmitter and receiver correlation. The correlation eigenvalues in the simulation in figure 1 are $[1.8, 0.2]$ for Tx and Rx correlation.

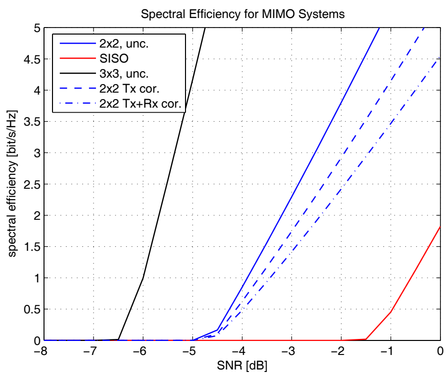

Fig. 1. Spectral Efficiency over $\frac{E_b}{N_0}$ for different MIMO systems and transmitter and receiver correlation for uninformed transmitter.

4.2. Informed transmitter

The ergodic capacity with an informed transmitter for $n_T \geq n_R$ is given by

$$C_{pCSI}(\rho) = \mathbb{E} \max_{\substack{\sum_{k=1}^{n_T} p_k = \rho \\ p_k \geq 0 \forall 1 \leq k \leq n_T}} \sum_{k=1}^{n_T} \log(1 + \lambda_k^H p_k). \quad (15)$$

Since, we are in the low power regime, there is always the complete power allocated to the largest eigenvalue of the instantaneous channel. We are in Rayleigh fading, the probability that two or more eigenvalues are equal and the largest is zero. This scenario corresponds to the case in [5, Theorem 12] in which the transmitter knows the eigenspace of the maximum eigenvalue of $\boldsymbol{H}^H\boldsymbol{H}$ but cannot employ temporal power control, i.e.

$$C_{pCSI}^0(\rho) = \mathbb{E}\log(1 + \rho\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H)). \quad (16)$$

The $\frac{E_b}{N_0}_{min}$ and the wideband slope S_0 are given by

$$\frac{E_b}{N_0}_{min}^{pCSI} = \frac{\log_e 2}{\mathbb{E}\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H)} \quad (17)$$

$$S_0^{pCSI} = \frac{2(\mathbb{E}\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H))^2}{\mathbb{E}(\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H))^2}. \quad (18)$$

The impact of correlation on the performance metrics in (17) and (18) is characterized in the following theorem.

Theorem 2 *With perfect CSI at the transmitter and receiver, the minimum $\frac{E_b}{N_0}$ is Schur-concave with respect to the channel correlation. The wideband slope is Schur-convex with respect to the correlation.*

This can be proven by showing that $\mathbb{E}\lambda_{max}(\boldsymbol{R}_T\boldsymbol{H}\boldsymbol{R}_R\boldsymbol{H}^H)$ decreases with increasing correlation in $\boldsymbol{R}_T$ and $\boldsymbol{R}_R$ and showing that $(\mathbb{E}\lambda_{max}(\boldsymbol{R}_T\boldsymbol{H}\boldsymbol{R}_R\boldsymbol{H}^H))^2$ increases slower with increasing correlation than $\mathbb{E}\lambda_{max}(\boldsymbol{R}_T\boldsymbol{H}\boldsymbol{R}_R\boldsymbol{H}^H)^2$.

Remark: Note, that a small $\frac{E_b}{N_0}$ is better than a large one. This means with less correlation a lower $\frac{E_b}{N_0}$ is achieved and correlation improves reliability. For the wideband slope it is the other way round: The higher $\mathcal{S}_0$ is the better is the performance of the systems.

4.3. Channel statistics knowledge

The ergodic capacity with covariance knowledge at the transmitter is given by [9, 10]

$$C_{covCSI}(\rho) = \max_{\substack{\sum_{k=1}^{n_T} p_k = \rho \\ p_k \geq 0 \forall 1 \leq k \leq n_T}} \mathbb{E}\log\left|\boldsymbol{I} + \sum_{k=1}^{n_T} p_k\lambda_k\tilde{\boldsymbol{w}}_k\tilde{\boldsymbol{w}}_k^H\right|$$

with independent Gaussian distributed $\tilde{\boldsymbol{w}}_k$ including the receive correlation matrix eigenvalues

$$\tilde{\boldsymbol{w}}_k = \left[\sqrt{\lambda_1^R}w_{1,k}, \sqrt{\lambda_2^R}w_{2,k}, \ldots, \sqrt{\lambda_{n_R}^R}w_{n_R,k}\right]^T.$$

For small SNR values, $p_1 = \rho$ and $p_2 = p_3 = \ldots = 0$. The ergodic capacity is the given by

$$C_{covCSI}^0(\rho) = \mathbb{E}\log\left(1 + \rho\lambda_1^T\sum_{k=1}^{n_R}\lambda_k^R w_k\right). \quad (19)$$

The first and second derivative of the capacity with respect to ρ at the point $\rho = 0$ are given by

$$\left(\frac{E_b}{N_0}\right)_{min}^{covCSI} = \frac{\log_e 2}{n_R\lambda_1^T} \quad (20)$$

$$\mathcal{S}_0^{covCSI} = \frac{2n_R^2}{\mathbb{E}\left(\sum_{k=1}^{n_R}\lambda_k^R w_k\right)^2}. \quad (21)$$

This correspond to the result in [5, equation (236)]. The following theorem characterizes the impact of correlation on the performance metrics in (20) and (21).

Theorem 3 *With covariance knowledge at the transmitter and perfect CSI at the receiver, the minimum $\frac{E_b}{N_0}$ is Schur-concave with respect to transmitter correlation and does not depend on the receiver correlation. The wideband slope is Schur-concave with respect to the receiver correlation and does not depend on the transmitter correlation.*

Proof: The Schur-concavity of the minimum $\frac{E_b}{N_0}$ is obvious, because the first eigenvalue of the transmit correlation λ_1^T is monotonic increasing with increasing correlation and its inverse therefore monotonic decreasing. The receive correlation occurs only in its sum of the eigenvalues. Since the trace is constrained to be equal to n_R, the receive correlation has no impact on the minimum $\frac{E_b}{N_0}$.

The transmit correlation does not occur in the wideband slope. It remains to show that the function $f(\boldsymbol{\lambda}^R) = \mathbb{E}\left(\sum_{k=1}^{n_R} \lambda_k^R w_k\right)^2$ is Schur-convex. In order to verify Schur's condition in (6), we obtain for the difference of the partial derivatives of f with respect to λ_1^R and λ_2^R the following expression

$$\begin{aligned} \Delta &= 2\mathbb{E}\left[(w_1 - w_2)\left(\sum_{k=1}^{n_R} \lambda_k^R w_k\right)\right] \\ &= 2\lambda_1^R \mathbb{E}(w_1^2 - w_1) - \lambda_2^R \mathbb{E}(w_2^2 - w_2) \\ &= \lambda_1^R - \lambda_2^R \geq 0. \end{aligned} \quad (22)$$

The inequality in (22) verifies Schur-condition. As a result, the wideband slope $\mathcal{S}_0$ as a function of receive correlation is Schur-concave.

Remark: Note, that here transmit correlation improves the system reliability, i.e. it decreases the minimum $\frac{E_b}{N_0}$. The more correlated the receive antennas are the less is the wideband slope $\mathcal{S}_0$, i.e. the performance decreases with increasing correlation.

4.4. Comparison between different CSI scenarios

The minimum $\frac{E_b}{N_0}$ of the three CSI scenarios are connected by the following inequalities

$$\log_e 2 \left(\frac{E_b}{N_0}_{min}^{nCSI}\right)^{-1} = n_R \leq n_R \lambda_1^T = \log_e 2 \left(\frac{E_b}{N_0}_{min}^{covCSI}\right)^{-1}$$

with equality for completely uncorrelated transmit antennas and

$$\log_e 2 \left(\frac{E_b}{N_0}_{min}^{covCSI}\right)^{-1} = n_R \lambda_1^T \leq$$

$$\mathbb{E}\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H) = \log_e 2 \left(\frac{E_b}{N_0}_{min}^{pCSI}\right)^{-1}$$

with equality for completely correlated transmit antennas and uncorrelated receive antennas. The last inequality follows from the fact that $\mathbb{E}\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H)$ is monotonic increasing with increasing transmit and receive correlation and from

$$\begin{aligned} &\mathbb{E}\lambda_{max}(\boldsymbol{H}\boldsymbol{H}^H) = \mathbb{E}\lambda_{max}(\boldsymbol{R}_T \boldsymbol{W}\boldsymbol{W}^H) \\ &\geq \lambda_{max}(\boldsymbol{R}_T)\mathbb{E}\lambda_{max}(\boldsymbol{W}\boldsymbol{W}^H) \geq \lambda_1^T n_R \end{aligned} \quad (23)$$

These inequalities are illustrated in figure 2. The left side corresponds to the completely correlated case $\lambda_2^T = 0$ and the right side corresponds to the completely uncorrelated case $\lambda_2^T = \lambda_1^T = 1$.

5. CONCLUSION

In this work, the spectral efficiency of MIMO systems in spatially correlated Rayleigh fading and with different types of CSI was studied. Depending on the type of CSI at the transmitter the minimum $\frac{E_b}{N_0}$ does or does not depend on the channel correlation. The connection between the dispersion of the channel as a measure of correlation and majorization has been pointed out. If the transmitter has no CSI transmit and receive correlation decrease the wideband slope. If the transmitter has perfect CSI, correlation improves the performance in terms of minimum $\frac{E_b}{N_0}$ and wideband slope $\mathcal{S}_0$. If the transmitter has covariance knowledge, the minimum $\frac{E_b}{N_0}$ and the wideband slope is Schur-concave, i.e. the first order behaviour of the spectral efficiency improves with increasing transmit correlation but the second order behavior degrades with increasing receiver correlation.

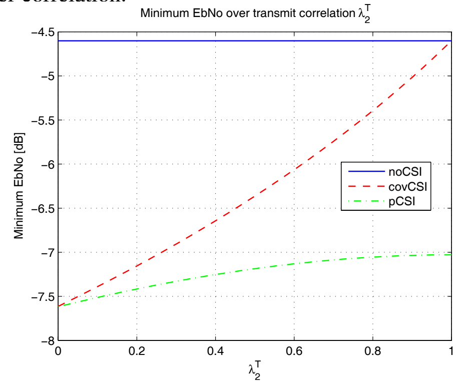

Fig. 2. Minimum $\frac{E_b}{N_0}$ over transmit correlation $\lambda_2^T = 2 - \lambda_1^T$, uncorrelated receive antennas.

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