D.O.A. ESTIMATION WITH SINGULAR NOISE CORRELATION MATRIX

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ABSTRACT

In this paper, a new method of direction of arrival (D.O.A.) estimation with environmental noise, whose spatial correlation matrix is singular, is proposed. In D.O.A. estimation, identification of signal and noise subspaces plays a very important role. The identification process can be achieved by (generalized) eigenvalue decomposition of the spatial correlation matrix of observations (with respect to that of noise), if these spatial correlation matrices are non-singular. However, these mathematical tools can not be applied to the problems in which the spatial correlation matrices are singular. The main idea of this work deeply depends on identification of proper and improper eigenvectors of the spatial correlation matrix of noise with respect to that of observations. The results of computer simulations are also presented to verify the efficacy of the proposed method.

1. INTRODUCTION

In D.O.A. estimation, represented by MUSIC [1] and ESPRIT [2], identification of signal and noise subspaces is essential. It is well known that when noise is spatially white, we can easily identify signal and noise subspaces by using eigenvalue decomposition of the spatial correlation matrix of observations, and when noise is spatially colored, we can also identify these subspaces by using generalized eigenvalue decomposition of the spatial correlation matrix of observations with respect to that of noise, if these correlation matrices are non-singular. However, these mathematical tools can not be applied to the problems that have singular correlation matrices.

In this paper, we propose a method of D.O.A. estimation that can be applied to the problems that have singular correlation matrices. The main idea of our method depends on identification of proper and improper eigenspaces [3] of the spatial correlation matrix of noise with respect to that of observations. The results of computer simulations are also presented to verify the efficacy of our method.

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2. PROBLEM FORMULATION

Let n, m $(< n)$, t, and ω be the number of observations (the number of microphones), the number of sound sources, frame and frequency indexes in short time Fourier domain. We assume that the observation vector $\boldsymbol{x}(\omega, t) \in \mathbf{C}^n$ is modeled as follows:

$$\boldsymbol{x}(\omega, t) = A(\omega)\boldsymbol{s}(\omega, t) + \boldsymbol{n}(\omega, t), \quad (1)$$

where $\boldsymbol{s}(\omega, t) \in \mathbf{C}^m$, $A(\omega) \in \mathbf{C}^{n\times m}$, and $\boldsymbol{n}(\omega, t) \in \mathbf{C}^n$ denote the sound source vector, the transfer function between sound sources and microphones, and noise vector. We also assume that $\boldsymbol{s}(\omega, t)$ and $\boldsymbol{n}(\omega, t)$ are zero-mean random variables and that they are mutually uncorrelated. Hereafter, we omit (ω, t) in notations, since there is no fear of confusion.

The objective of D.O.A. estimation is to identify the direction where the sound sources exist, incorporating identification of signal and noise subspaces by using the series of observations $\boldsymbol{x}$ and statistical properties of $\boldsymbol{n}$.

3. SUMMARY OF D.O.A. ESTIMATION

In this section, we summarize D.O.A. estimation, especially in cases that the spatial correlation matrix of noise is non-singular.

Let $R_{\boldsymbol{s}}$ and Q be the spatial correlation matrices of the sound sources $\boldsymbol{s}$ and noise $\boldsymbol{n}$ written by

$$E[\boldsymbol{s}\boldsymbol{s}^*] = R_{\boldsymbol{s}}, \ \ E[\boldsymbol{n}\boldsymbol{n}^*] = Q, \quad (2)$$

then, the correlation matrix of observations $\boldsymbol{x}$ is written by

$$R_{\boldsymbol{x}} = E[\boldsymbol{x}\boldsymbol{x}^*] = R + Q, \quad (3)$$

where $R = AR_{\boldsymbol{s}}A^*$. We assume $\mathrm{rank}(A) = \mathrm{rank}(R_{\boldsymbol{s}}) = m$ for all following discussions.

When $Q = \sigma^2 I_n$ (including the case of $\sigma^2 = 0$) is satisfied, identification of signal and noise subspaces is achieved by eigenvalue decomposition of $R_{\boldsymbol{x}}$ written by

$$R_{\boldsymbol{x}} = P_1(\Lambda_1 + \sigma^2 I_n)P_1^*, \quad (4)$$

where I_n, P_1 and Λ_1 denote the identity matrix of degree n, the unitary matrix consisting of the eigenvectors of $R_{\boldsymbol{x}}$ and the diagonal matrix consisting of the eigenvalues of $R_{\boldsymbol{x}}$. In this case, because of the non-negative definiteness of R, the signal subspace is spanned by the eigenvectors corresponding to the eigenvalues larger than σ^2, and the noise subspace is spanned by those corresponding to the eigenvalues equal to σ^2.

When Q is non-singular matrix which may be non-diagonal, we consider

$$\boldsymbol{y} = Q^{-1/2}\boldsymbol{x}. \tag{5}$$

The spatial correlation matrix of $\boldsymbol{y}$ is written by

$$\begin{array}{rcl} R_{\boldsymbol{y}} = E[\boldsymbol{y}\boldsymbol{y}^*] & = & Q^{-1/2}R_{\boldsymbol{x}}Q^{-1/2} \\ & = & Q^{-1/2}RQ^{-1/2} + I_n. \end{array} \tag{6}$$

Here, let

$$Q^{-1/2}RQ^{-1/2} = P_2\Lambda_2P_2^* \tag{7}$$

be the eigenvalue decomposition of $Q^{-1/2}RQ^{-1/2}$, then Eq.(6) is reduced to

$$R_{\boldsymbol{x}}(Q^{-1/2}P_2) = Q(Q^{-1/2}P_2)(\Lambda_2 + I_n). \tag{8}$$

Eq.(8) means that the diagonal elements of $(\Lambda_2 + I_n)$ and the column vectors of $(Q^{-1/2}P_2)$ denote the eigenvalues and the eigenvectors of the generalized eigenvalue problem written by

$$R_{\boldsymbol{x}}\boldsymbol{u} = \lambda Q\boldsymbol{u}. \tag{9}$$

Therefore, the signal subspace is spanned by the eigenvectors corresponding to the eigenvalues larger than 1, and the noise subspace is spanned by the eigenvectors corresponding to the eigenvalue 1 in this case. Note that Eq.(8) can be transformed to

$$Q(Q^{-1/2}P_2) = R_{\boldsymbol{x}}(Q^{-1/2}P_2)(\Lambda_2 + I_n)^{-1}. \tag{10}$$

Eq.(10) means that the diagonal elements of $(\Lambda_2 + I_n)^{-1}$ and the column vectors of $(Q^{-1/2}P_2)$ denote the eigenvalues and the eigenvectors of the generalized eigenvalue problem written by

$$Q\boldsymbol{u} = \lambda R_{\boldsymbol{x}}\boldsymbol{u}. \tag{11}$$

In terms of Eq.(10), the signal subspace is spanned by the eigenvectors corresponding to eigenvalues smaller than 1.

Let $X_1 \in C^{n\times(n-m)}$ be the matrix consisting of the eigenvectors that span the noise subspace, then D.O.A. estimation using MUSIC (for instance) is reduced to finding the θ that gives a peak of the criterion

$$J_1(\theta) = \frac{\boldsymbol{a}(\theta)^*\boldsymbol{a}(\theta)}{\boldsymbol{a}(\theta)^*X_1X_1^*\boldsymbol{a}(\theta)}, \tag{12}$$

where $\boldsymbol{a}(\theta)$ denotes a weight vector for microphones, that is fixed by the direction θ.

4. D.O.A. ESTIMATION FOR SINGULAR CORRELATION MATRICES

As is mentioned in the previous section, when environmental noise exists, identification of signal and noise subspaces deeply depends on non-singularity of the spatial correlation matrix of noise. Therefore, when the spatial correlation matrix of noise is singular, we can not use a mathematical tool such as generalized eigenvalue decomposition for identification of signal and noise subspaces. In this section, we show that these subspaces can be identified by proper and improper eigenspaces [3], even if the spatial correlation matrices are singular. We also apply the method of identification of the subspaces to D.O.A. estimation using MUSIC.

Firstly, we introduce mathematical tools which play important roles in the following discussions.

Theorem 1 *Let A and B be n.n.d. Hermitian matrices of degree n, then there exists a non-singular matrix T that makes T^*BT and T^*AT be diagonal matrices [3].*

Note that T^*BT can be written by the following matrix of degree n, if $\mathrm{rank}(B) = r$ is satisfied [3].

$$T^*BT = I_{n,r} \equiv \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \tag{13}$$

Definition 1 *Let A and B be a Hermitian matrix and an n.n.d. Hermitian matrix. The scalar λ and the vector $\boldsymbol{w}$ are called "proper eigenvalue" and "proper eigenvector" of A with respect to B, when*

$$A\boldsymbol{w} = \lambda B\boldsymbol{w}, \quad B\boldsymbol{w} \neq \boldsymbol{0} \tag{14}$$

holds. On the other hand the vector $\boldsymbol{w}$ is called "improper eigenvector", when

$$A\boldsymbol{w} = B\boldsymbol{w} = \boldsymbol{0}$$

holds [3].

On the basis of Theorem 1, let T be the non-singular matrix that makes T^*RT and T^*QT be diagonal matrices, then,

$$\begin{array}{l} Q = (T^*)^{-1}I_{n,r}T^{-1}, \quad R = (T^*)^{-1}\Lambda_3T^{-1} \\ R_{\boldsymbol{x}} = (T^*)^{-1}(\Lambda_3 + I_{n,r})T^{-1} \end{array} \tag{15}$$

holds with $\mathrm{rank}(Q) = r$, where Λ_3 denotes a diagonal matrix.

Theorem 2 *Let $I_{n,r}$ and Λ_3 be the matrices described in Eq.(15), then,*

$$\mathcal{R}(I_{n,r}) \subset \mathcal{R}(\Lambda_3 + I_{n,r}) \tag{16}$$

holds.

Proof

Let $R = P_4 \Lambda_4 P_4^*$ be the eigenvalue decomposition of R, then

$$P_4 \Lambda_4 P_4^* = (T^*)^{-1}(\Lambda_3) T^{-1}$$

holds, and it can be transformed to

$$\Lambda_3 = T^* P_4 \Lambda_4 P_4^* T = (\Lambda_4^{1/2} P_4^* T)^* (\Lambda_4^{1/2} P_4^* T), \quad (17)$$

since diagonal elements of Λ_4 are real non-negative values. On the basis of Eq.(17), it is concluded that the diagonal elements of Λ_3 are also non-negative values, since the right-hand side of Eq.(17) is obviously an *n.n.d.* matrix.

If a vector $\boldsymbol{x}$ belongs to the null space of $(\Lambda_3 + I_{n,r})$, written by $\mathcal{N}(\Lambda_3 + I_{n,r})$, then

$$\boldsymbol{x}^* \Lambda_3 \boldsymbol{x} = 0, \ \ \boldsymbol{x}^* I_{n,r} \boldsymbol{x} = 0,$$

holds, since Λ_3 is a non-negative diagonal matrix. It immediately follows that $(I_{n,r}\boldsymbol{x})^*(I_{n,r}\boldsymbol{x}) = 0$, which concludes

$$\mathcal{N}(\Lambda_3 + I_{n,r}) \subset \mathcal{N}(I_{n,r}),$$

that is identical to Eq.(16). □

From Eq.(15) and Theorem 2, it immediately follows that

$$QT = R_{\boldsymbol{x}} T (\Lambda_3 + I_{n,r})^+ I_{n,r} \quad (18)$$

holds. Eq.(18) gives the solution of the proper (and improper) eigenvalue problem written by

$$Q\boldsymbol{w} = \lambda R_{\boldsymbol{x}} \boldsymbol{w}, \ \ \mathcal{R}_{\boldsymbol{x}} \boldsymbol{w} \neq \boldsymbol{0}.$$

Therefore, the column vector of T written by $\boldsymbol{t}_i$, that satisfies $R_{\boldsymbol{x}} \boldsymbol{t}_i = \boldsymbol{0}$, corresponds to the improper eigenvector of Q with respect to $R_{\boldsymbol{x}}$, since $Q\boldsymbol{t}_i = \boldsymbol{0}$ is also satisfied that is easily derived from $\mathcal{R}(Q) \subset \mathcal{R}(R_{\boldsymbol{x}})$. On the other hand, $\boldsymbol{t}_i$, that does not satisfy $R_{\boldsymbol{x}} \boldsymbol{t}_i = \boldsymbol{0}$, corresponds to the proper eigenvectors of Q with respect to $R_{\boldsymbol{x}}$, and the corresponding diagonal elements of $(\Lambda_3 + I_{n,r})^+ I_{n,r}$ represent the proper eigenvalues of Q with respect to $R_{\boldsymbol{x}}$. The proper eigenvectors, corresponding to the proper eigenvalues 1, span the noise subspace, and those, corresponding to the proper eigenvalues smaller than 1, span the signal subspace.

Identification of signal and noise subspaces in this situation seems to be able to be reduced to the generalized eigenvalue problem written by

$$R_{\boldsymbol{x}} \boldsymbol{u} = \lambda (TT^*)^{-1} \boldsymbol{u}, \quad (19)$$

since Eq.(15) is easily transformed to

$$R_{\boldsymbol{x}} T = (TT^*)^{-1} T (\Lambda_3 + I_{n,r}). \quad (20)$$

However, we should not use the solution of Eq.(20), since we can not discriminate the eigenvector corresponding to the signal subspace from that corresponding to the noise subspace, when the diagonal element of $(\Lambda_3 + I_{n,r})$ is 1.

Let X_2 be the matrix consisting of the improper eigenvectors and proper eigenvectors that span the noise subspace, then D.O.A. estimation using MUSIC is reduced to finding the θ that gives a peak of the criterion

$$J_2(\theta) = \frac{\boldsymbol{a}(\theta)^* \boldsymbol{a}(\theta)}{\boldsymbol{a}(\theta)^* X_2 X_2^* \boldsymbol{a}(\theta)}, \quad (21)$$

where $\boldsymbol{a}(\theta)$ denotes the same vector used in the previous section. Note that we use not only the proper eigenvectors corresponding to the noise subspace but also improper eigenvectors for constructing the matrix X_2, so that, the criterion $J_2(\theta)$ does not have peaks at the subspace that neither the signal nor noise exists.

5. COMPUTER SIMULATIONS

In this section, we show the results of computer simulations in order to verify the efficacy of the proposed method. Figure 1 shows the layout of microphones, a sound source, and an environmental noise source. We assume that the signal and the noise sources are far enough from the microphones. The sampling rate is $32kHz$, the sound source is written by

$$s(t) = \sin(2\pi f t), \ \ (f = 500Hz),$$

and Gaussian white noise, whose variances are $\sigma^2 = 1.0$, 100.0, are adopted for the noise. Note that Q is singular, since the same noise is observed by all microphones. Although the conditions for the noise might be so special, these conditions make the work of our method clear. We investigated the performance of MUSIC using the method of identification of signal and noise subspaces based on normal eigenvalue decomposition (as the competitor) and our method. Note that we can not adopt generalized eigenvalue decomposition for the competitor, since it can not be applied to this problem that has a singular noise correlation matrix.

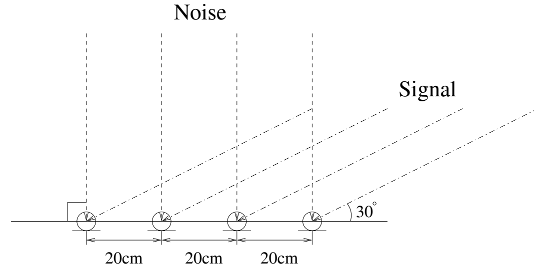

Fig. 1. Layout of microphones, sound sources, and noise.

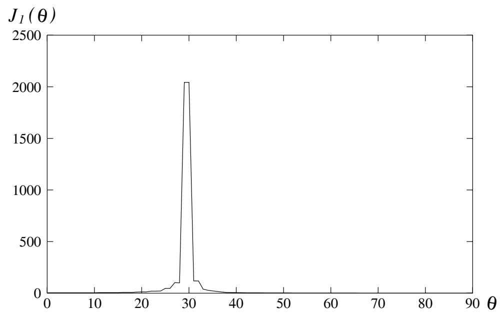

Fig. 2. Results of D.O.A. estimation by normal eigenvalue decomposition ($\sigma^2 = 1.0$).

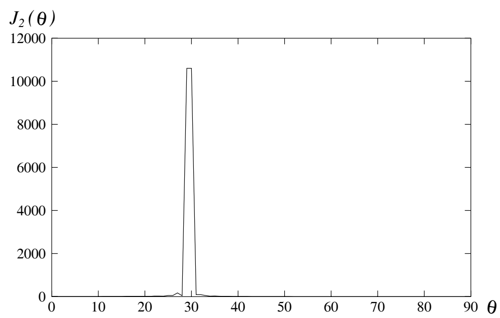

Fig. 3. Results of D.O.A. estimation by the proposed method ($\sigma^2 = 1.0$).

5.1. Results for $\sigma^2 = 1.0$

Figures 2 and 3 show the results of D.O.A. estimation at $500Hz$ using the methods of identification of signal and noise subspaces based on normal eigenvalue decomposition and the proposed method in the case of $\sigma^2 = 1.0$.

Under these conditions, the eigenvector corresponding to the maximum eigenvalue in normal eigenvalue decomposition of the spatial correlation matrix of observations, is correctly spanning the signal subspace. Therefore, both of methods correctly estimated the direction of arrival of the sound source.

5.2. Results for $\sigma^2 = 100.0$

Figures 4 and 5 show the results as the same with the previous subsection in the case of $\sigma^2 = 100.0$.

Under these conditions, the eigenvector corresponding to the maximum eigenvalue in normal eigenvalue decomposition of the spatial correlation matrix of observations, catches the noise subspace, since the variance of noise is comparatively large. Therefore, it is confirmed that normal eigenvalue decomposition is useless in such conditions, while the proposed method works well.

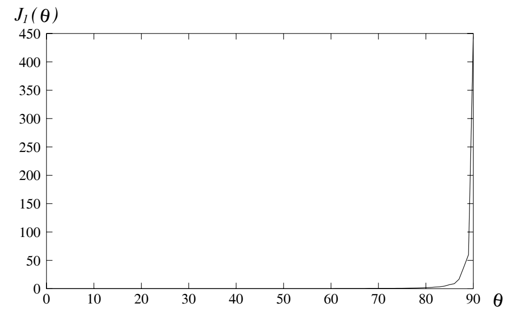

Fig. 4. Results of D.O.A. estimation by normal eigenvalue decomposition ($\sigma^2 = 100.0$).

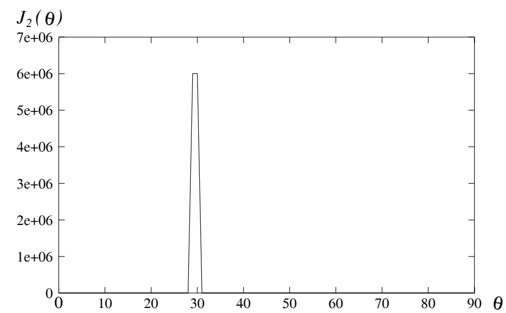

Fig. 5. Results of D.O.A. estimation by the proposed method ($\sigma^2 = 100.0$).

6. CONCLUSION

In this paper, we proposed a method of D.O.A. estimation that can be applied to the problems that have singular spatial correlation matrices, incorporating proper and improper eigenspaces of the spatial correlation matrix of noise with respect to that of observations. The efficacy of the proposed method is confirmed by the results of computer simulations.

7. REFERENCES

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