AN INTEGRO-DIFFERENTIAL METHOD FOR ADAPTIVE FILTERING OF ADDITIVE OR MULTIPLICATIVE NOISE

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ABSTRACT

In this paper, we present a new adaptive filtering method of either the additive noise or the multiplicative noise. The proposed method is stated with a differential equation of temporal evolution of the problem of interest. It achieves an improvement of the efficiency of the well-known iterated Lee filter in the spatial domain. Mainly, it incorporates the local determination of the optimal regions which are subsequently used to estimate the different local statistics involved in the filtering method. This estimation is carried out differently according to the nature of the processed pixel. An adapted decisional criteria indicates if the pixel belongs either to a contour or to an homogeneous zone, following the idea used in the classical anisotropic methods. Then, the efficiency of the proposed method, for which we can prove existence and uniqueness of the solution, is assessed on several images degraded artificially. The results are compared to the main used filters in order to confirm the theoretical waitings.

1. INTRODUCTION

The numerical images are often interfered by perturbation phenomena either due to the experimental conditions during the acquisition or to the acquisition system itself. These perturbations generate degradations in the observed image that penalize the subsequent processing operations such as segmentation, analysis and its interpretation. It is therefore necessary to perform a pre-treatment on the observed image in order to best reconstruct the original image. In this paper, we only consider the most common cases of filtering, in other words, the restoration of an image degraded by either an additive noise, or a multiplicative noise. The problem is difficult to resolve because a particular luminous intensity at an image pixel, can give rise to an infinite number of possible combination of the original image and the noise. More, noisy images exhibit a strong variability from one pixel to another. An observed local intensity variation might be due either to the noise or to an edge. Obviously, edges are significant features in images as they carry important information for the subsequent image processing tasks. They have to be recovered without signal distortion.

Following the idea developed by the classical anisotropic methods, we propose to characterize each pixel as belonging either to a contour or an homogeneous zone. However, the great difference with these methods is that we use integral operators instead of spatial derivatives for filtering. In all the anisotropic methods, the pixels of contour are implicitly distinguished from the others by using the gradient. We can mention the total variation based filters [12] [5], the Perona-Malik model [14] [4] [7] [16] and the Shock-filters [13] [2]. In the method of Alvarez and *al* [1], the pixels of contour are simply detected with the help of a contrast threshold. The main difficulty consists on exactly determining this contrast threshold. Unfortunately, it is not unique but depends on the luminous intensity of the reference pixel. A previous study has been realized about this subject by Chehdi and *al* [6]. It led to the visual perception function or function of the eye sensitivity to contrasts. This function gives in an adaptive manner the value of the contrast threshold to distinguish two pixels according to their luminous intensity. The method we propose involves a soft-switching edge detection scheme to examine whether a local intensity variation corresponds to an edge or not, followed by invoking proper combined traditional filtering operations. The detection of contour is achieved by means of an operator that thresholds the gradient magnitude of the smoothed image by the value of the visual perception function at the considered pixel. The detection performance is not related to the accurate determination of several thresholds. The paper is organized as follows: In section 2, we briefly recall the theory presented by Lee [10]. In section 3 we give the reader a better understanding of the proposed filter. We precise the key lines to be used to show the existence and the uniqueness of the solution of the considered differential equation. In section 4, analysis of the results on a well-known real data artificially corrupted highlights the efficacy of the approach. We quantitatively evaluated image quality after spatial filtering by the well-known criterion : *the mean absolute error* (MAE), *the mean square error* (MSE) and *the maximal error* (ME) and *the Peak Signal-to-Noise-Ratio* (PSNR$_{dB}$). After a last refinement of the proposed method, we are concluding in section 5.

2. THE LEE FILTER

In the optimal methods, we can mention the filter presented by Lee, whose performances have been evaluated. In order to attenuate the noise while preserving the contours, Lee proposes an optimal filter completely determined by its *gain* $\mathbb{K}$. If we note v the original image and u_0 the observed image on a compact support Ω, then the observation equation for the additive noise hypothesis, is given by :

$$u_0 = v + \mathrm{b} \tag{1}$$

where b is an additive noise non correlated with v such that its mean value $\mathbb{E}[\mathrm{b}] = 0$ and its variance $\mathrm{var}[\mathrm{b}] = \sigma^2$.

For the multiplicative noise hypothesis, if we note u_∞ the

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original image, then the observation equation writes :

$$u_0 = u_\infty \times \mathrm{n} \tag{2}$$

where n is a multiplicative noise, non correlated with the original image u_∞, and such that $\mathbb{E}[\mathrm{n}] = 1$ and $\mathrm{var}[\mathrm{n}] = \sigma_\mathrm{n}^2$. Kuan proposes in [9] a pseudo-observation equation to adapt the Lee additive filter to the multiplicative model:

$$u_0 = u_\infty + \mathrm{b}' = u_\infty + u_\infty \times (\mathrm{n} - 1) \tag{3}$$

where b' is the additive pseudo-observation noise, non correlated with u_∞ but dependent of u_∞ such that $\mathbb{E}[\mathrm{b}'] = 0$ and $\mathrm{var}[\mathrm{b}'] = \mathbb{E}[u_\infty^2]\mathrm{var}[\mathrm{n}]$.

In the two cases, the filter is completely determined by *the gain* $\mathbb{K}$:

$$\mathbb{F}_{\mathrm{Lee}}[u_0] \triangleq \mathbb{E}[u_0] + \mathbb{K}\,(u_0 - \mathbb{E}[u_0])\,. \tag{4}$$

The statistics involved in the calculation of the local gain are estimated from pixels belonging to the neighborhood. As a result, the precised selection of pixels is all the more important for the performance of the filter. To take into account the local configuration of the image and to preserve the contours, the gain must be locally calculated from a window with a reduced size $R \times R$ centered on (i, j) the pixel to process. In the following, a pixel will be noted (i, j) in the discrete case, and (x, y) in the continuous case. The Lee filter presents the main shortcoming to calculate the local statistics on predefined masks. On the contrary, in the filtering methods based on partial differential equations, the pixels belonging to contours are differently processed from the other pixels.

3. INTEGRO-DIFFERENTIAL STATEMENT OF THE ITERATED LEE FILTER

3.1. Evolution equation

We call *evolution equation of the iterated Lee filter* the differential equation defined by the following equation.

$$\begin{aligned}&\dot{u}(t) + u(t) = \mathbb{P}[u(t), t] \triangleq \\ &\Big(1 - \mathbb{K}_\Phi[u(t), t]\Big)\mathbb{E}_\Phi[u(t), t] + \mathbb{K}_\Phi[u(t), t]\; u(t)\end{aligned} \tag{5}$$

where $\dot{u}$ is the temporal derivative of u and $\mathbb{P}[u, t]$ is a perturbation operator of the identity operator [8]. If we note $\mathbb{E}_\Phi[u, t]$ and $\mathrm{var}_\Phi[u, t]$ the local mean and the local variance, then *the local gain* $\mathbb{K}_\Phi[u, t]$ for the additive model is defined by:

$$\mathbb{K}_\Phi[u, t] \triangleq \left(1 - \frac{\sigma(t)^2}{\mathrm{var}_\Phi[u, t]}\right)^+ \tag{6}$$

For the multipicative model, the gain is defined by:

$$\mathbb{K}_\Phi[u, t]^\times \triangleq \left(1 - \frac{\mathbb{E}[u^2, t]}{\mathrm{var}u, t}\,\frac{\sigma_\mathrm{n}(t)^2}{1 + \sigma_\mathrm{n}(t)^2}\right)^+ \tag{7}$$

The variance of the noise is explicitly used in the calculation of the gain. So, its evaluation during successive iterations is an important factor for the performance of the filter. We wish to improve the basic understanding of the image filtering process and to determine how it is affected by the noise. The following modeling 8 is proposed : The effective variance of noise at the time instant t is given by :

$$\sigma(t) \triangleq \left[\mathrm{TV}(u_0)^{-1}\mathrm{TV}(u(t))\right]\sigma \tag{8}$$

where TV is the total variation norm and such that $g{*}u$ is as closed as possible to u. As the evolution equation provides a smoothing filter, the TV-norm of the solution is probably decreasing. So we make the assumption that the standard deviation of the noise is decreasing proportionally to the TV-norm. We now consider the *iterated sequence of Lee* corresponding to the successive iterations of the Lee filter on the observed image u_0 defined by $u^{n+1} = \mathbb{F}_{\mathrm{Lee}}[u^n]$ and $u^0 = u_0$. If we firstly assume that the temporal discretization step takes the particular unit value $\Delta t = 1$, then we can write the recurrence relation of the iterated Lee sequence under the following form:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \Big(1 - \mathbb{K}[u^n]_{i,j}\Big)\left(\mathbb{E}[u^n]_{i,j} - u_{i,j}^n)\right) \tag{9}$$

where $\mathbb{K}[u^n]_{i,j}$ is the local gain of the image u^n and $\mathbb{E}[u^n]_{i,j}$ the local mean computed within the selected Lee mask centered in (i, j).

3.2. Local and Adaptive Determination of the Supports

As precised above, the proposed improvement consists on determining more relevant local support used to compute the local statistics according to the nature of the pixels. We can model the local support of any local statistics calculation by an operator $\Phi[u, t]$ called the *local statistical kernel* that depends on the image u and the abstract time t. This operator is choosen such that $\Phi[u, t](x, y; x', y')$ is equal to 1 if (x', y') belongs to the local support $K(x, y)$ and vanishes if (x', y') is sufficiently far to $K(x, y)$. Now, following the idea introduced before by Alvarez and *al* in [1], we define the *total local statistical kernel* $\Phi[u, t]$ of the proposed method:

$$\Phi[u, t] \triangleq \Theta[u, t]\Phi_{1|2}[u, t]\tau[W] + \Big(1 - \Theta[u, t]\Big)\tau[w] \tag{10}$$

where the operator $\Theta[u, t]$ and $\Phi_{1|2}[u, t]$ are defined by :

$$\Theta[u, t](x, y) \triangleq H\left(\int_\Omega \Phi_{1|2}[u, t](x, y; x', y')\mathrm{d}x'\mathrm{d}y' - L\right), \tag{11}$$

$$\Phi_{1|2}[u, t] \triangleq \Psi[u]\Phi_1[u] + \Big(1 - \Psi[u]\Big)\Phi_2[u, t] \tag{12}$$

where w (resp. W) is a truncation function on the square $[-r/2, +r/2]$ (resp. $[-R/2, +R/2]$). Then, $\tau[w]$ (resp. $\tau[W]$) is the translatory motion operator of w (resp. W) to the current position (i.e. $\tau[w](x, y; x', y') = w(x - y, x' - y')$). The use of $\tau[w]$ was introduced by Lee [11] for the 2σ filter. The underlying idea is to adapt the kernel $\Phi[u, t]$ in order to eliminate the isolated pixels. This occurs when the number of selected pixels in the window of size $R \times R$ is lower than some threshold L in accordance with the 2σ filter. This threshold depends of course on the size of the window : $L < 4$ if $R = 7$, $L < 3$ if $R = 5$, $L = 1$ if $R = 3$. $\Phi_{1|2}[u, t]$ realize through the edge detector $\Psi[u]$ the switching between the local statistical kernels $\Phi_1[u]$ and $\Phi_2[u, t]$ defined below. They have been used separately in the past: the first one $\Phi_1[u]$ for pixels belonging to contours, the second one $\Phi_2[u, t]$ for the pixels of the homogeneous zone.

The *average selected smoothing technique filter* described by Asano and *al* [3], allows to disregard the pixels $(x, y) \in \Omega$ in the considered local window whose contrast value $|u(x, y) - u(x', y')|$ is lower to the magnitude value of the discontinuity $|\vec{\nabla} u(x, y)|$.

In other words, this filter enables to only take into account pixels whose contrast value is lower to the discontinuity magnitude. $\Phi_1[u]$ is then given by :

$$\Phi_1[u](x,y;x',y') \triangleq \\ H\Big(|u_\sigma(x',y') - u_\sigma(x,y)| - |\vec{\nabla} u_\sigma(x,y)|\Big)\,. \tag{13}$$

Therefore, it correctly enhances the contours but does not smooth sufficiently in the homogeneous zones.

The *adaptive averaging filter* described by Pomalaza-Raes and *al* in [15] only depends on the variance of the noise. For each pixel $(x,y) \in \Omega$, we calculate the mean average value of the luminous intensity of pixels $(x',y') \in \Omega$ belonging to the considered local window whose difference $|u(x,y) - u(x',y')|$ is lower to a threshold C. $\Phi_2[u,t]$ is defined by :

$$\Phi_2[u,t](x,y;x',y') \triangleq \\ H\Big(|u_\sigma(x',y') - u_\sigma(x,y)| - C[u,t](x,y)\Big)\Big). \tag{14}$$

with $C[u,t](x,y) = S\Big(2\sigma(t), f(u_\sigma(x,y))\Big)$ where S is a smooth approximation of the supremum function. This filter designed to eliminate additive noise is used in the 2σ filter developed by Lee in [11]. It performs a good smoothing in the homogeneous zones but sometimes introduces some distortions close to the contours.

In the proposed method, we decide to combine the two local statistical kernels $\Phi_1[u]$ and $\Phi_2[u,t]$ through the selective processing of the contour pixels and the other pixel. This is carried out through the *contour detection operator* :

$$\Psi[u](x,y) \triangleq H\Big(|\vec{\nabla} u_\sigma(x,y)| - f(u_\sigma(x,y))\Big) \tag{15}$$

where we denote f the regular approximation of the visual perception function. It returns a unit value if the pixel (x,y) is a contour pixel and zero otherwise. The operator $\Psi[u]$ allows to control the switching between the filtering of a contour pixel and the filtering of an homogeneous pixel.

Note that the local statistics $\mathbb{E}_\Phi[u,t]$, $\mathrm{var}_\Phi[u,t]$ in (5) are calculated with the total local statistical kernel $\Phi[u,t]$.

The analysis of the evolution equation (5) allows us to prove the existence and uniqueness of its solution. The definition of the local statistical kernel Φ corresponds to a combination of integral operators T_Φ. If $\Phi \in C^2 \cap W^{2,\infty}(\mathbb{R}^{m+n} \times \Omega^2, \mathbb{R})$ then $\mathrm{T}_\Phi \in C^1(\mathcal{D}(\mathrm{T}_\Phi), \mathrm{L}^\infty(\Omega,\mathbb{R}))$, is Lipschitz and bounded on the centered balls, where $W^{s,p}$ are the Sobolev spaces. This result insures us that the perturbation satisfies three hypotheses such that to find a solution of (5) is equivalent to find a fixed point of the sequence $\mathcal{U}$:

$$\mathcal{U}[u](t) \triangleq \mathrm{e}^{-t}\left(u_0 + \int_0^t \mathrm{e}^s \mathbb{P}[u(s),s]\mathrm{d}s\right). \tag{16}$$

Then, if $u \in C^1(\mathbb{R}^+, \mathrm{L}^\infty(\Omega,\mathbb{R}))$ is a solution of the problem (16) then it satisfies :

$$\forall t \in \mathbb{R}^+, [\underline{u}(t), \overline{u}(t)] \subset [\underline{u}_0, \overline{u}_0]. \tag{17}$$

where $\overline{u} = \sup\mathrm{ess}\, u$ and $\underline{u} = \inf\mathrm{ess}\, u$. This is a direct consequence of one of the hypothesis satisfied by the perturbation. The problem (16) admits a unique solution $u \in C^1(\mathbb{R}^+, \mathrm{L}^\infty(\Omega,\mathbb{R}))$. The demonstration follows the same scheme than that of the theorem of Cauchy-Lipschitz-Picard. We seek to show the convergence of a sequence toward a fixed point where the Lipschitz constant is obtained according to one of the hypothesis satisfied by the perturbation and is independent of the current iteration.

However, we can refine the computation of the support by considering masks which better translate the local structure of the image for pixels belonging to contours. The idea is to make use of the level line passing by the processed pixel defining the mask. Thus, we can obtain a fine structure that is composed of pixels, the selection of which contributes to minimize the variance along the line. Then, we apply the masks close to contours and keep applying the previous thresholding based mask in the homogeneous zones [Lee^+].

4. EXPERIMENTAL RESULTS

Experiments were carried out on different images. Among them, we select the real image [boat] for its very thin details (*fig.* 1). The degraded image has been processed by the classical Lee filter [Lee^1], the iterated Lee filter [Lee^2], the improved Lee filter [Lee^3], the TV Filter [TV] [5], and the refined Lee filter [Lee^+]. The respective values of the four criteria, (MAE), (MSE), (ME), (PSNR_{dB}) while applying each retained filtering methods on the two previous images are regrouped in table (*tab.* 1) and (*tab.* 2). The results of the refined method [Lee^+] are only presented for the additive noise hypothesis. From a subjective viewpoint (visual observation) the images processed by the improved and refined iterated filter seem to show a better quality (*fig.* 1), (*fig.* 2). We can easily observe less residues, especially close to the contours. For the classic or iterated Lee filter, we can on the other hand observe the main default of this kind of filters by pointing at the penalyzing effects induced by the use of predefined masks. Obviously, the models can not correspond exactly with the local configuration of the image, what translates by some destructive effects even though the local gain introduced by Lee allows to limit these artifacts. From an objective viewpoint, the modifications proposed for the Lee filter seem to be efficient. The obtained values for the different criterion generally confirm the visual impression with globally lower errors for the proposed method.

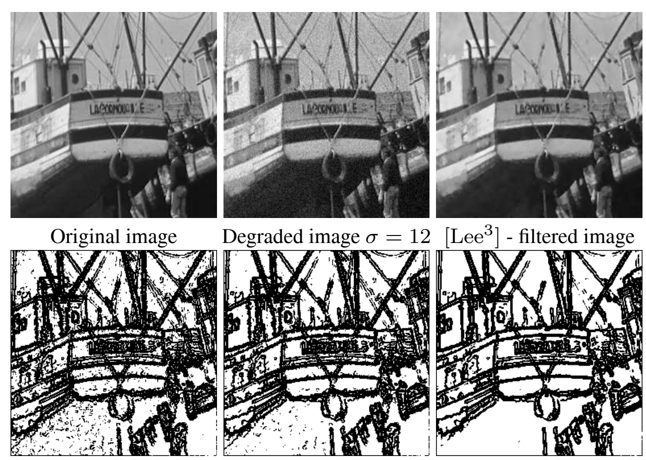
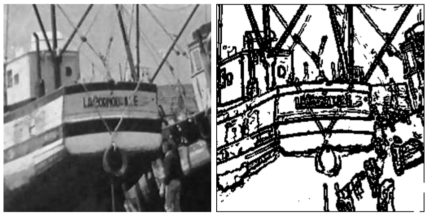


Fig. 1. Original, Degraded, Filtered Images - Edge detection on the iterations of [Lee^3] noise filtering on the $\sigma = 12$ noisy image [boat]

For the additive noise hypothesis, the results of the last refinement of the method are promising and slightly improve quality of the finally obtained image.



$[\text{Lee}^+]$ edge detection

Fig. 2. Filtering of the $\sigma = 12$ Noisy Image [boat] by the $[\text{Lee}^+]$ Filter

	[ME]	[MAE]	[MSE]	$[\text{PSNR}_{\text{dB}}]$
$\sigma = 8$	38	6.382	8.019	30.048
$[\text{Lee}^1]$	35	4.419	5.900	32.714
$[\text{Lee}^2]$	35	4.278	5.77	32.907
[TV]	80	4.115	5.679	33.046
$[\text{Lee}^3]$	**32**	3.883	5.208	33.798
$[\text{Lee}^+]$	33	**3.824**	**5.114**	**33.955**
$\sigma = 10$	48	7.962	9.994	28.136
$[\text{Lee}^1]$	53	5.189	7.019	31.205
$[\text{Lee}^2]$	43	4.980	6.81	31.468
[TV]	80	4.750	6.463	31.922
$[\text{Lee}^3]$	**37**	4.508	6.107	32.415
$[\text{Lee}^+]$	**37**	**4.475**	**6.006**	**32.559**
$\sigma = 12$	57	9.514	11.933	26.595
$[\text{Lee}^1]$	53	5.874	7.981	30.090
$[\text{Lee}^2]$	53	5.605	7.741	30.355
[TV]	75	5.485	7.351	30.804
$[\text{Lee}^3]$	**41**	5.074	6.898	31.357
$[\text{Lee}^+]$	47	**5.041**	**6.763**	**31.528**
$\sigma = 14$	67	11.051	13.858	25.297
$[\text{Lee}^1]$	59	6.564	8.927	29.117
$[\text{Lee}^2]$	59	6.199	8.576	29.465
[TV]	97	6.389	8.481	29.561
$[\text{Lee}^3]$	**57**	5.682	7.697	30.405
$[\text{Lee}^+]$	**57**	**5.605**	**7.507**	**30.621**

Table 1. Additive Noise Filtering on the Image [boat]

5. CONCLUSION

The method proposed in this paper presents an original statement of the iterated Lee filter. It leads to the writing of a temporal evolution equation as a differential equation for which we prove the existence and the uniqueness of the solution. To take into account the local configuration of the image and to preserve fine structures of the image, it first determines a relevant estimate of the support used to calculate the local statistics introduced in the differential equation. This is achieved by differentiating the pixels according to their nature, either belonging to a contour or a homogeneous zone following the idea developed with the classic anisotropic methods. The obtained filter selects locally between a selective filter that properly enhances the contours but does not sufficiently smooth in the homogeneous zones and an adaptive filter, that conversely performs a good smoothing in the homogeneous zones but sometimes introduces some distortions close to the contours. The selective switching according the nature of the pixel is done through the operator of contour detection. The efficiency of the method is checked in simulation by comparison to the well-known benchmarked filters.

	[ME]	[MAE]	[MSE]	$[\text{PSNR}_{\text{dB}}]$
$\sigma = 0.1$	73	9.353	13.284	25.664
$[\text{Kuan}^1]$	82	5.759	8.327	29.721
$[\text{Kuan}^2]$	95	5.512	8.035	30.031
$[\text{Kuan}^3]$	**68**	**5.018**	**7.284**	**30.883**
$\sigma = 0.2$	155	18.096	25.503	19.999
$[\text{Kuan}^1]$	**96**	9.259	13.175	25.736
$[\text{Kuan}^2]$	97	8.740	12.603	26.122
$[\text{Kuan}^3]$	**96**	**7.996**	**11.643**	**26.81**
$\sigma = 0.3$	224	26.074	36.709	16.835
$[\text{Kuan}^1]$	135	12.782	18.372	22.847
$[\text{Kuan}^2]$	**112**	12.002	17.326	23.357
$[\text{Kuan}^3]$	115	**11.487**	**16.758**	**23.646**

Table 2. Multiplicative Noise Filtering on the Image [boat]

6. REFERENCES

[1] L. Alvarez, P.-L. Lions, J.-M. Morel, "Image Selective Smoothing an Edge Detection by Nonlinear Diffusion (ii)," *SIAM J. Numer. Anal.*, Vol. 39, No. 3, pp. 845-866, June 1992.

[2] filtrelee L. Alvarez, L. Mazorra, "Signal and image restoration using schock filters and anisotropic diffusion," *SIAM J. Numer. Anal.*, Vol. 31, No. 2, pp. 590-605, April 1994.

[3] T. Asano, N. Yokoga, "Image segmentation scheme for low level computer vision," *Pattern Recognition*, 1981.

[4] F. Catte, T. Coll, P. L. Lions, J. L. Morel, "Image selective smoothing and edge detection by nonlinear diffusion (i)," *SIAM J. Numer. Anal.*, Vol. 29, No. 1, pp. 183-193, Februray 1992.

[5] T. F. Chan and S. Osher and J. Shen, "The Digital TV Filter and nonlinear denoising," *IEEE Trans. on Image Processing*, Vol. 10, No. 2, pp. 231-241, Febr. 2001.

[6] K. Chehdi, C. Cariou and C. Kermad, "Image segmentation and texture classification using local thresholds and 2D AR modeling," in *European Signal Processing Conference*, pp. 30-33, 1994.

[7] G. H. Cottet, M. El Ayyadi, "A Volterra type model for image processing," *IEEE Trans. on Image Processing*, Vol. 7, No. 3, March 1998.

[8] T. Kato, "Perturbation Theory for Linear Operators," *Springer-Verlag, Classics in Mathematics,* 1976.

[9] D. T. Kuan, "Adaptive Noise Smoothing Filter for Images with Signal Dependent Noise," *IEEE Trans. on Pattern Analysis and Machine Intel.*, Vol. PAMI-7, No. 2, pp. 165-177, March 1985.

[10] J. S. Lee, "Refined Filtering of Image Noise using Local Statistics," *Computer Graphics and Image Processing*, Vol. 15, pp. 380-389, 1981.

[11] J. S. Lee, "Digital Image Smoothing and the Sigma Filter," *Computer Graphics and Image Processing*, No. 24, pp. 259-269, 1983.

[12] A. Marquina, S. Osher, "A new time dependant model based on level set motion for nonlinear deblurring and noise removal," *Lecture Notes in Computer Science*, Vol. 1682, pp. 429-434, 1999.

[13] S. Osher, L. I. Rudin "Feature-oriented image enhancement using schock filters," *SIAM J. Numer. Anal.*, Vol. 27, No. 4, pp. 919-940, Aug. 1990.

[14] P. Perona, J. Malik, "Scale space and edge detection using anisotropic diffusion," *IEEE Trans. on Pattern Analysis and Machine Intel.*, Vol. 12, No. 7, pp. 629-639, 1990.

[15] C. A. Pomalaza-Raes, C. D. McGillem, "An Adaptive Nonlinear Edge Preserving Filter," *IEEE Trans. Acoust. Speech, Signal Processing, ASSP*, Vol. 32, No. 3, pp. 571-576, 1984.

[16] J. Weickert, B. M. ter Haar Romeny, M. A. Viergever "Efficient and reliable Schemes for nonlinear diffusion filtering," *IEEE Trans. on Image Processing*, Vol. 7, No. 3, pp. 398-410, March 1998.