THREE-DIMENSIONAL FAST ALGORITHM SOLUTION FOR OCTANT-BASED THREE-DIMENSIONAL YULE-WALKER EQUATIONS

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ABSTRACT

This paper presents an extension of the solution for two-dimensional (2-D) Yule-Walker equations, useful for linear prediction (LP) parameter estimation to the three-dimensional (3-D) case. The resulting fast recursive 3-D algorithm has a significant computational advantage over direct solution of the 3-D Yule-Walker equations because it exploits the triply-Toeplitz structure.

1. INTRODUCTION

There have been previous attempts to extend the two-dimensional (2-D) estimation approach to the three-dimensional (3-D) case [5], [6]. This paper presents the techniques for estimating 3-D autoregressive (AR) parameters from 3-D autocorrelation sequence (ACS) values using 3-D Yule-Walker equations with a recursive solution operating directly in the 3-D octant-space. This computationally simple and fast performing algorithm of a close heritage to the original 2-D algorithm, as will be shown, by a new approach involving recursive estimation of a related set of triply Toeplitz block matrices, from which the octant-space AR parameters can be solved.

2. THREE-DIMENSIONAL AUTOREGRESSIVE PARAMETER MATRICES

2.1. Three-Dimensional Autoregressive Process

A 3-D autoregressive sequence $x[i,j,k]$ is generated by driving a 3-D linear shift-invariant filter with a 3-D white noise sequence $w[i,j,k]$,

$$x[i,j,k] = -\sum_m \sum_n \sum_l a[l,m,n]\, x[i-l, j-m, k-n] + w[i,j,k] \quad (1)$$

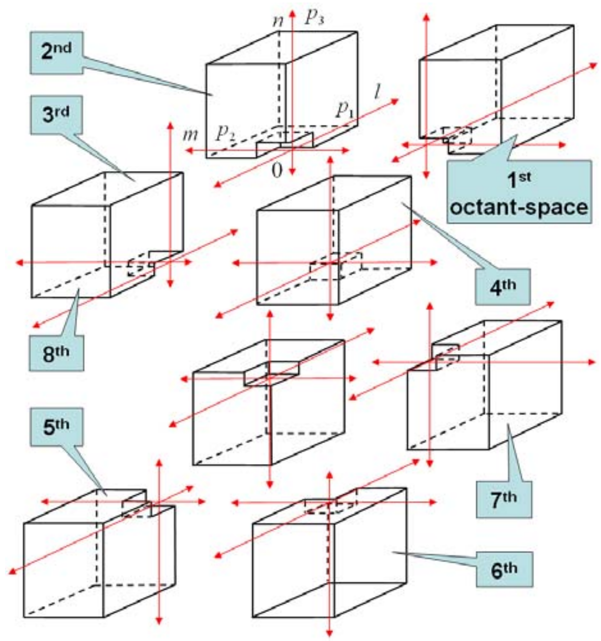

Fig. 1. Typical eight octant-space regions of support for 3-D AR parameter arrays

Fig.1 illustrates the regions of octant support for eight octant-space AR arrays. A 3-D linear prediction estimate of the array sample $x[l,m,n]$ will have the form

$$\hat{x}[l,m,n] = -\sum_m \sum_n \sum_l a[l,m,n]\, x[i-l, j-m, k-n]$$

in which $a[l,m,n]$ is a 3-D linear prediction/AR co-efficients. If an octant region of support is selected, the 3-D linear prediction coefficients that minimize the variance of the error $x[l,m,n] - \hat{x}[l,m,n]$,

$$\rho_{LP} = \rho_{white} = \mathrm{E}\left\{ \left| x[l,m,n] - \hat{x}[l,m,n] \right|^2 \right\} \quad (2)$$

will yield a linear prediction error that is a 3-D whitening

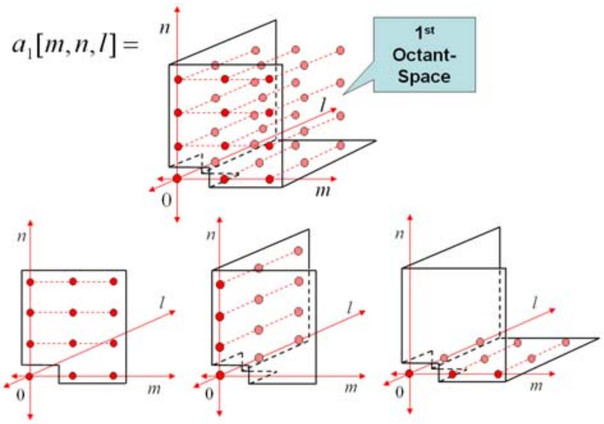

Fig. 2. 2x3x2 AR parameters in the first-octant space

process if the 3-D LP is of the same 3-D order as the 3-D AR process. The region of support for the first-, second-, third-, fourth-, fifth-, sixth-, seventh-, and eighth octant-space AR parameter arrays $a^i[l,m,n]$ (i = 1,2,3,4,5,6,7 and 8) are defined as,

$$
\begin{array}{ll}
a_1[l,m,n] & o \le m \le p_2 \;\; if \;\; 1 \le n \le p_3 \; , 1 \le m \le p_2 \;\; if \;\; n=0 \\
 & o \le m \le p_2 \;\; if \;\; 1 \le l \le p_1 \; , 1 \le m \le p_2 \;\; if \;\; l=0 \\
 & o \le n \le p_3 \;\; if \;\; 1 \le l \le p_1 \; , 1 \le n \le p_3 \;\; if \;\; l=0 \\
a_2[l,m,n] & -p_2 \le m \le 0 \;\; if \;\; 1 \le n \le p_3 \; , -p_2 \le m \le 1 \;\; if \;\; n=0 \\
 & -p_2 \le m \le 0 \;\; if \;\; 1 \le l \le p_1 \; , -p_2 \le m \le -1 \;\; if \;\; l=0 \\
 & o \le n \le p_3 \;\; if \;\; 1 \le l \le p_1 \; , 1 \le n \le p_3 \;\; if \;\; l=0 \\
a_3[l,m,n] & -p_2 \le m \le 0 \;\; if \;\; 1 \le n \le p_3 \; , -p_2 \le m \le -1 \;\; if \;\; n=0 \\
 & -p_2 \le m \le 0 \;\; if \;\; -p_1 \le l \le -1 \; , -p_2 \le m \le -1 \;\; if \;\; l=0 \\
 & o \le n \le p_3 \;\; if \;\; -p_1 \le l \le -1 \; , 1 \le n \le p_3 \;\; if \;\; l=0 \\
a_4[l,m,n] & o \le m \le p_2 \;\; if \;\; 1 \le n \le p_3 \; , 1 \le m \le p_2 \;\; if \;\; n=0 \\
 & o \le m \le p_2 \;\; if \;\; -p_1 \le l \le -1 \; , 1 \le m \le p_2 \;\; if \;\; l=0 \\
 & o \le n \le p_1 \;\; if \;\; -p_1 \le l \le -1 \; , 1 \le n \le p_3 \;\; if \;\; l=0 \\
 & \vdots \\
a_8[l,m,n] & o \le m \le p_2 \;\; if \;\; -p_3 \le n \le -1 \; , 1 \le m \le p_2 \;\; if \;\; n=0 \\
 & o \le m \le p_2 \;\; if \;\; 1 \le l \le p_1 \; , 1 \le m \le p_2 \;\; if \;\; l=0 \\
 & -p_3 \le n \le 0 \;\; if \;\; 1 \le l \le p_1 \; , -p_3 \le n \le -1 \;\; if \;\; l=0
\end{array}
$$

2.2. Three-Dimensional Yule-Walker Equations

The 3-D Yule-Walker equations for a 3-D AR process are obtained by multiplying Eq. (1) by $x^*[i-l, j-m, k-n]$ and taking the expectation to yield

$$\sum_m \sum_n \sum_l a[l,m,n] \, r_{xx}[i-l, j-m, k-n] = \begin{Bmatrix} \rho_\omega & for[l,m,n]=[0,0,0] \\ 0 & otherwise \end{Bmatrix}.$$

The summation ranges can be selected to be any one of the 8 octants of $a^i[l,m,n]$. In anticipation of the fast computational algorithm to be presented, we shall assume

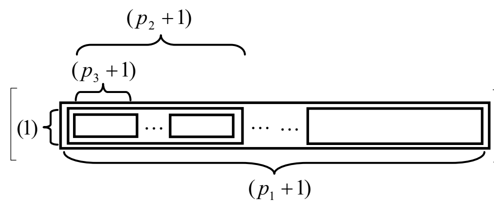

Fig. 3. Dimension of super block vector

that subscript *p1p2p3* means *p1*, the point on the '*l*' axis, is a variable order parameter. Then, the *p2* point on the '*m*' axis and the *p3* point on the '*n*' axis in Fig. 2 are assumed to be fixed order parameters. The 3-D Yule-Walker equations for the support regions can be arranged, or ordered, into at least six convenient super block vector forms

$$\underline{\underline{\mathbf{a}}}^i_{p1p2p3}, \underline{\underline{\mathbf{a}}}^i_{p1p3p2}, \underline{\underline{\mathbf{a}}}^i_{p2p1p3}, \underline{\underline{\mathbf{a}}}^i_{p2p3p1}, \underline{\underline{\mathbf{a}}}^i_{p3p1p2}, \textit{ and } \underline{\underline{\mathbf{a}}}^i_{p3p2p1}$$

by ordering the 3-D AR coefficients. An alternative block vector representation of the first octant-space Yule-Walker equation is

$$
\begin{aligned}
\underline{\underline{\rho}}^1_{p1p2p3} &= \mathrm{E}\left\{ \underline{\underline{\mathbf{x}}}_{p1p2p3} \, \underline{\underline{\mathbf{x}}}^{\mathbf{H}}_{p1p2p3} \right\} \underline{\underline{\mathbf{a}}}^1_{p1p2p3} \\
&= \underline{\underline{\mathbf{R}}}_{p1p2p3} \underline{\underline{\mathbf{a}}}^1_{p1p2p3} \qquad (3)
\end{aligned}
$$

The *super block vector* $\underline{\underline{\mathbf{a}}}_{p1p2p3}$ has a superscript 1. It designates this as a set of the first octant AR parameters. It is composed of (p_1+1) *block vectors*, each of dimension (p_2+1)(p_3+1) x 1,

$$\underline{\underline{\mathbf{a}}}^i_{p1p2p3} = [\underline{\mathbf{a}}^1_{p1p2p3}[0] \; \underline{\mathbf{a}}^1_{p1p2p3}[1] \; \cdots \; \underline{\mathbf{a}}^1_{p1p2p3}[p_1]]^{\mathsf{T}}$$

which is defined in terms of the *block vectors*

$$\underline{\mathbf{a}}^1_{p1p2p3}[l] = [\mathbf{a}^1_{p1p2p3}[l,0] \; \mathbf{a}^1_{p1p2p3}[l,1] \; \cdots \; \mathbf{a}^1_{p1p2p3}[l,p_2]]^{\mathsf{T}}$$

which is also defined in terms of the *scalar elements*

$$
\begin{aligned}
&\mathbf{a}^1_{p1p2p3}[l,m] \\
&\quad = [a^1_{p1p2p3}[l,m,0] \; a^1_{p1p2p3}[l,m,1] \; \cdots \; a^1_{p1p2p3}[l,m,p_3]]^{\mathsf{T}}
\end{aligned}
$$

The *super block vector* $\underline{\underline{\rho}}_{p1p2p3}$ has p_1 numbers of zero *block vector* $\underline{0}_1$ and one top super block entry $\underline{\rho}^1_{p1p2p3}$. The *block vector* $\underline{\rho}_{p1p2p3}$ has all zero entries, except for the top entry, which is the noise variance ρ_{white}. Note that '**0**' is vector of (p_3+1) zeros and $\underline{\mathbf{0}}$ is a column block vector of (p_2+1)(p_3+1) zeros.

$$\underline{\underline{\rho}}^1_{p1p2p3} = [\underline{\rho}^1_{p1p2p3} \;\; \underline{\mathbf{0}} \cdots \underline{\mathbf{0}}]^{\mathsf{T}}$$

which is defined in terms of the *block vectors*

$$\underline{\rho}^1_{p1p2p3} = [\rho^1_{p1p2p3} \;\; \mathbf{0} \cdots \mathbf{0}]^{\mathsf{T}}$$

which is in turn defined in terms of the *scalar elements*

$$\rho^1_{p1p2p3} = [\rho^1_{white} \;\; 0 \cdots 0]^{\mathsf{T}}$$

The data vector is also a super block vector of super block dimension (p_1+1).

$$\underline{\underline{\mathbf{x}}}_{p1p2p3}[l,m,n] = [\underline{\underline{\mathbf{x}}}_{p1p2p3}[k]\ \underline{\underline{\mathbf{x}}}_{p1p2p3}[k-1] \cdots \underline{\underline{\mathbf{x}}}_{p1p2p3}[k-p_1]]$$

Super block matrix $\underline{\underline{\mathbf{R}}}_{p1p2p3}$ has dimension (p_1+1)(p_2+1)(p_3+1)x(p_1+1)(p_2+1)(p_3+1). From Eq. (3), we can derive $\underline{\underline{\mathbf{R}}}_{p1p2p3}$ as,

$$E\left\{\underline{\underline{\mathbf{x}}}_{p1p2p3}\,\underline{\underline{\mathbf{x}}}^{\mathsf{H}}_{p1p2p3}\right\}=$$

$$E\left\{\begin{bmatrix} x\,[l,m,n] \\ \vdots \\ x\,[l,m,n-p_3] \\ x\,[l,m-1,n] \\ \vdots \\ x\,[l,m-1,n-p_3] \\ \vdots \\ \vdots \\ x\,[l,m-p_2,n-p_3] \\ \vdots \\ \vdots \\ \vdots \\ \underline{x\,[l-p_1,m-p_2,n-p_3]} \end{bmatrix} \bullet \begin{bmatrix} x\,[l,m,n] \cdots x\,[l,m,n-p_3]\ x\,[l,m-1,n] \cdots \\ x\,[l,m-p_2,n-p_3] \cdots\cdots x\,[l-1,m-p_2,n-p_3] \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots x\,[m-p_1,n-p_2,l-p_3] \vdots \end{bmatrix}\right\}$$

Each *Super block* $\underline{\mathbf{R}}_{p1p2p3}$ is (p_2+1)x(p_2+1) *block* Toeplitz (each block element is Toeplitz) and each Toeplitz block is (p_3+1)x(p_3+1) scalar elements.

$$\underline{\underline{\mathbf{R}}}_{p1p2p3} = \begin{bmatrix} \underline{\mathbf{R}}_{p1p2p3}[\ 0\ ] & \cdots & \underline{\mathbf{R}}_{p1p2p3}[p_1] \\ \vdots & & \vdots \\ \underline{\mathbf{R}}_{p1p2p3}[-p_1] & \cdots & \underline{\mathbf{R}}_{p1p2p3}[\ 0\ ] \end{bmatrix}$$

$$\underline{\mathbf{R}}_{p1p2p3}[0] =$$

$$\begin{bmatrix} \begin{bmatrix} r[\ 0,\ 0,\ 0] & r[\ 0,\ 0,\ 1] & \cdots & r[\ 0,\ 0,\ p_3] \\ r[\ 0,\ 0,-1] & & & \vdots \\ \vdots & & & \vdots \\ r[\ 0,0,-p_3] & \cdots & \cdots & r[\ 0,\ 0,\ 0] \end{bmatrix} & \cdots & \begin{bmatrix} r[\ 0,\ p_2,\ 0] & r[0,p_2,1] & \cdots & r[0,\ p_2,p_3] \\ r[\ 0,-p_2-1] & & & \vdots \\ \vdots & & & \vdots \\ r[0,-p_2,-p_3] & \cdots & \cdots & r[0,\ p_2,0] \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} r[0,-p_2,\ 0] & r[0,p_2,\ 1] & \cdots & r[\ 0,\ p_2,p_3] \\ r[0,-p_2,-1] & & & \vdots \\ \vdots & & & \vdots \\ r[0,-p_2,-p_3] & \cdots & \cdots & r[\ 0,-p_2,0] \end{bmatrix} & \cdots & \begin{bmatrix} r[\ 0,\ 0,\ 0] & r[\ 0,\ 0,\ 1] & \cdots & r[0,\ 0,\ p_3] \\ r[\ 0,\ 0,\ -1] & & & \vdots \\ \vdots & & & \vdots \\ r[\ 0,\ 0,-p_3] & \cdots & \cdots & r[0,\ 0,\ 0] \end{bmatrix} \end{bmatrix}$$

Therefore, matrix $\underline{\underline{\mathbf{R}}}_{p1p2p3}$ is said to be *triply Toeplitz* or *super block Toeplitz*. A subscript *p1p2p3* is used to remind the reader that '*variable order p1*' and *'fixed order p2 and p3'* ordering has been used.

3. FAST SOLUTION OF 3-D NORMAL EQUATIONS

If the 3-D autocorrelation sequence is known, then it will be could be shown that the first octant-space and eighth octant-space parameters satisfy the following 3-D Yule-Walker normal equations

$$\underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}\underline{\underline{\mathbf{R}}}_{p1p2p3} = \underline{\underline{\rho}}^{1}_{p1p2p3} \quad \textbf{and} \quad \underline{\underline{\mathbf{a}}}^{8}_{p1p2p3}\underline{\underline{\mathbf{R}}}_{p1p2p3} = \underline{\underline{\rho}}^{8}_{p1p2p3}$$

which has the alternative representation

$$\underline{\underline{\mathbf{a}}}^{i}_{p1p2p3}\underline{\underline{\mathbf{R}}}_{p1p2p3} = [\ \underline{\mathbf{P}}^{i}_{p1p2p3}\ \underline{\mathbf{0}} \cdots\cdots \underline{\mathbf{0}}\ ] \qquad (4)$$

It can be shown that $\underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}$ and $\underline{\underline{\mathbf{a}}}^{5}_{p1p2p3}$ are obtained as complex conjugates of $\underline{\underline{\mathbf{a}}}^{2}_{p1p2p3}$ and $\underline{\underline{\mathbf{a}}}^{6}_{p1p2p3}$, $\underline{\underline{\mathbf{a}}}^{3}_{p1p2p3}$ and $\underline{\underline{\mathbf{a}}}^{7}_{p1p2p3}$,and the last set $\underline{\underline{\mathbf{a}}}^{4}_{p1p2p3}$ and $\underline{\underline{\mathbf{a}}}^{8}_{p1p2p3}$. Autocorrelation symmetry properties are

$$r[\ l,\ m,\ n] = r^{*}[-l,-m,-n] \qquad r[\ l,-m,\ n] = r^{*}[-l,\ m,-n]$$
$$r[-l,-m,\ n] = r^{*}[\ l,\ m,-n] \qquad r[-l,\ m,\ n] = r^{*}[\ l,-m,-n]$$

Fig. 4 illustrates the two coefficients (i = 1 and 5) which have hermitian symmetry property. A fast computational algorithm for solution of $\underline{\underline{\mathbf{a}}}^{i}_{p1p2p3}$ is not based on direct solution for the 3-D AR parameters, but is based on solving a special variant of the 3-D AR algorithm involving the solution of the following set of 3-D normal equation of order p and index k. Based on the property $\underline{\underline{\mathbf{a}}}_{p1p2p3} = \underline{\mathbf{J}}\,\underline{\underline{\mathbf{a}}}^{*}_{p1p2p3}\underline{\mathbf{J}}$, we can form

$$\left(\underline{\mathbf{J}}\,\underline{\underline{\mathbf{a}}}^{*}_{p1p2p3}\underline{\mathbf{J}}\right)\underline{\underline{\mathbf{R}}}_{p1p2p3} = [\underline{\mathbf{0}} \cdots\cdots \underline{\mathbf{0}}\ \ \underline{\mathbf{J}}\,\underline{\mathbf{P}}^{*}_{p1p2p3}\underline{\mathbf{J}}]$$

where

$$\underline{\mathbf{R}}_{p}[l,m,n] = \sum_{k=0}^{p} \underline{\mathbf{x}}_{p}[k]\ \underline{\mathbf{x}}^{\mathsf{H}}_{p}[k] + \underline{\mathbf{J}}\,\underline{\mathbf{x}}^{*}_{p}[k]\ \underline{\mathbf{x}}^{\mathsf{T}}_{p}[k]\,\underline{\mathbf{J}}$$

and the block vectors of block dimension (p_2+1)(p_3+1)x(p_2+1)(p_3+1) are defined as

$$\underline{\underline{\mathbf{a}}}_{p} = [\ \underline{\mathbf{I}}\ \ \underline{\mathbf{A}}_{p}\ [1] \cdots\cdots \underline{\mathbf{A}}_{p}\ [p]\ ]$$

Therefore, we can say that a related block linear prediction matrix relationship is

$$[\ \underline{\mathbf{I}}\ \ \underline{\mathbf{A}}_{p1p2p3}[1] \cdots \underline{\mathbf{A}}_{p1p2p3}[p_1]\,\underline{\underline{\mathbf{R}}}_{p1p2p3} = [\ \underline{\mathbf{P}}_{p1p2p3}\ \underline{\mathbf{0}} \cdots\cdots \underline{\mathbf{0}}\ ]$$

In which $\underline{\mathbf{I}}$ is a (p_2+1)(p_3+1) x (p_2+1)(p_3+1) identity matrix, the block linear prediction parameter matrixes $\underline{\mathbf{A}}_{p1p2p3}\ [p_1]$ for 1≤ k ≤p_1 and block linear prediction covariance matrix $\underline{\mathbf{P}}_{p1p2p3}$ have dimension (p_2+1)(p_3+1)x (p_2+1)(p_3+1).

Note that at, $\underline{\mathbf{P}}_{p}$ =$\underline{\mathbf{P}}_{p1p2p3}$ and $\underline{\mathbf{R}}_{p}$ = $\underline{\mathbf{R}}_{p1p2p3}$ in Eq. (3), so one derives $\underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}$ from $\underline{\underline{\mathbf{a}}}_{p}$ as follows

$$\underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}[0] = [\ \underline{\mathbf{I}}\ \ \underline{\mathbf{0}} \cdots\cdots \underline{\mathbf{0}}\ ][\underline{\mathbf{P}}_{p1p2p3}]^{-1}$$

and scaled such that $a^{1}[0,\ 0,\ 0] = 1$, as follows

$$\underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}[k] = \underline{\underline{\mathbf{a}}}^{1}_{p1p2p3}[0]\ \underline{\mathbf{A}}^{1}_{p1p2p3}[k] \qquad \textit{for } 1 \le k \le p_1$$

Similarly

$$\underline{\underline{\mathbf{a}}}^{8}_{p1p2p3}[0] = [\ \underline{\mathbf{0}} \cdots\cdots \underline{\mathbf{0}}\ \ \underline{\mathbf{I}}\ ][\underline{\mathbf{P}}_{p1p2p3}]^{-1}$$

also scaled such that $a^{8}[0,\ 0,\ 0] = 1$

$$\underline{\underline{\mathbf{a}}}^{8}_{p1p2p3}[k] = \underline{\underline{\mathbf{a}}}^{8}_{p1p2p3}[0]\ \underline{\mathbf{A}}^{8}_{p1p2p3}[k] \qquad \textit{for } 1 \le k \le p_1$$

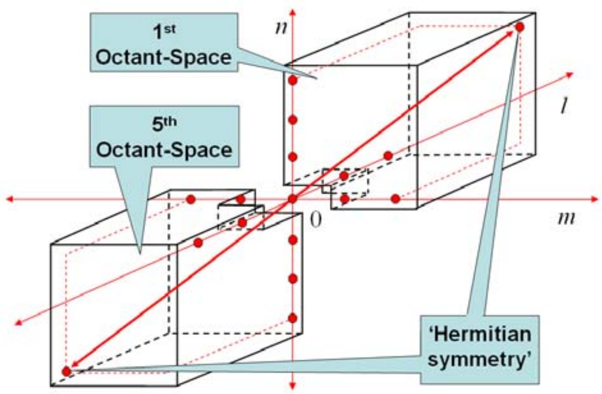

Fig. 4. Illustrate of the complex conjugates

4. RECURSIVE SOLUTION FOR 3-D SUPER BLOCK LINEAR PREDICTION PARAMETER MATRICES

Since $\underline{\underline{\mathbf{R}}}_p = \underline{\underline{\mathbf{R}}}_p^{\mathbf{H}}$ is super block Toeplitz (triply Toeplitz), we can show that the 3-D autocorrelation matrix is hermitian and is centrosymmetric $\underline{\mathbf{R}} = \underline{\mathbf{J}}\underline{\mathbf{R}}^{*}\underline{\mathbf{J}}$. Reflection matrix $\underline{\mathbf{J}}$ has $(p_2+1)(p_3+1)\mathrm{x}(p_2+1)(p_3+1)$ dimension. The Triply Toeplitz structure of can be exploited to develop the 3-D version of the recursive 1-D Levinson algorithm that solves Eq. (4). This paper already presented Eq. (4) and it may alternatively be expressed as Eq. (5) Using the centrosymmetric property and the identity matrix $\underline{\mathbf{I}}$, we can find the 3-D reflection coefficient matrix $\underline{\mathbf{K}}_{p+1}$, such that the following expression is valid

$$\begin{aligned}&[\,\underline{\mathbf{I}}\ \ \underline{\mathbf{A}}_{p+1}[1]\cdots\cdots\underline{\mathbf{A}}_{p+1}[p]\ \ \underline{\mathbf{A}}_{p+1}[p+1]]\\&\quad=[\,\underline{\mathbf{I}}\ \ \underline{\mathbf{A}}_{p}[1]\cdots\cdots\underline{\mathbf{A}}_{p}[p]\ \ \underline{\mathbf{0}}\,]\\&\qquad+\underline{\mathbf{K}}_{p+1}[\,\underline{\mathbf{0}}\ \ \underline{\mathbf{J}}\ \underline{\mathbf{A}}_{p}^{*}[p]\underline{\mathbf{J}}\cdots\cdots\underline{\mathbf{J}}\ \underline{\mathbf{A}}_{p}^{*}[1]\underline{\mathbf{J}}\ \ \ \underline{\mathbf{I}}\,]\end{aligned}\tag{5}$$

if we multiply both sides on the right by at order, this will yield

$$\begin{aligned}&[\underline{\mathbf{P}}_{p+1}\ \ \underline{\underline{\mathbf{0}}}\ \cdots\ \underline{\mathbf{0}}\ \ \underline{\mathbf{0}}\,]\\&\quad=[\underline{\mathbf{P}}\ \ \ \underline{\mathbf{0}}\ \ \cdots\ \ \underline{\mathbf{0}}\ \ \underline{\Delta}_{p+1}]\\&\qquad+\underline{\mathbf{K}}_{p+1}[\,\underline{\mathbf{J}}\ \underline{\Delta}_{p+1}^{*}\underline{\mathbf{J}}\ \ \ \underline{\mathbf{0}}\cdots\cdots\underline{\mathbf{0}}\ \ \underline{\mathbf{J}}\underline{\mathbf{P}}_{p}^{*}\underline{\mathbf{J}}\,]\end{aligned}\tag{6}$$

where

$$\underline{\Delta}_{p+1} = \underline{\underline{\mathbf{R}}}_p[p+1] + \sum_{k=1}^{p}\sum_{j=1}^{p}\underline{\mathbf{A}}_k[j]\underline{\underline{\mathbf{R}}}_p[k+1-j] \tag{7}$$

We can use the useful properties $\underline{\Delta} = \underline{\mathbf{J}}\underline{\Delta}^{\mathbf{H}}\underline{\mathbf{J}}$ from the original Levinson algorithm. Also, Eg. (5) will be balanced if we select

$$\begin{aligned}\underline{\mathbf{K}}_{p+1} &= \underline{\mathbf{A}}_{p+1}[p+1] = -\underline{\Delta}_{p+1}(\underline{\mathbf{J}}\ \underline{\mathbf{P}}^{*}\ \underline{\mathbf{J}})^{-1}\\&= -\underline{\Delta}_{p+1}\underline{\mathbf{J}}\,(\underline{\mathbf{P}}^{*})^{-1}\underline{\mathbf{J}}\end{aligned}\tag{9}$$

which creates the following order-update recursion

$$\underline{\mathbf{A}}_{p+1}[k] = \underline{\mathbf{A}}_{p}[k] + \underline{\mathbf{K}}_{p+1}(\underline{\mathbf{J}}\ \underline{\mathbf{A}}_{p}^{*}[p+1-k]\underline{\mathbf{J}}) \tag{10}$$

from (6) and (9), it is possible to derive the following recursion of the covariance matrix

$$\underline{\mathbf{P}}_{p+1} = (\underline{\mathbf{I}} - \underline{\mathbf{K}}_{p+1}[\underline{\mathbf{J}}\,\underline{\mathbf{K}}_{p+1}^{*}\underline{\mathbf{J}}])\underline{\mathbf{P}}_{p} = \underline{\mathbf{P}}_{p}(\underline{\mathbf{I}} - [\underline{\mathbf{J}}\,\underline{\mathbf{K}}_{p+1}^{\mathbf{T}}\underline{\mathbf{J}}])\underline{\mathbf{K}}_{p+1}^{\mathbf{H}} \tag{11}$$

This reduces the computational burden of general three-dimensional algorithm. Furthermore, this fast procedure permits computation of the AR parameters for all eight spaces simultaneously.

5. CONCLUSION

In this work, we have presented an efficient implementation of the 3-D Yule-Walker equations. The closed-form expression of the inverse matrix enables further simplifications of the 3-D coefficients due to the highly structured problem formulation. This work has shown us the possibility of decreasing the computational complexity as compared with the classical approach, especially for larger matrix sizes. Furthermore, we can develop, in future study, a novel 3-D lattice algorithm to estimate the forward prediction matrices based on the 3-D Yule-Walker equation that will be useful in producing a spectral estimation with high frequency resolution.

6. REFERENCES

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