PARAMETRIC ESTIMATION OF MULTI-DIMENSIONAL AFFINE TRANSFORMATIONS: AN EXACT LINEAR SOLUTION

Rami Hagege and Joseph M. Francos

Elec. & Comp. Eng. Dept.
Ben-Gurion University
Beer Sheva 84105, Israel

ABSTRACT

We consider the general framework of planar object recognition based on a set of known templates. While the set of templates is known, the tremendous set of possible affine transformations that may relate the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. Given an observation on one of the known objects, subject to an unknown affine transformation of it, our goal is to estimate the deformation that transforms some pre-chosen representation of this object (template) into the current observation. The direct approach for estimating the transformation is to apply each of the deformations in the affine group to the template in search for the deformed template that matches the observation. We propose a method that employs a set of non-linear operators to replace this high dimensional problem by an *equivalent linear problem*, expressed in terms of the unknown affine transformation parameters. This solution is further extended to include the case where the deformation relating the observed signature of the object and the template is composed both of the geometric deformation due to the affine transformation of the coordinate system and a constant illumination change. The proposed solution is *unique and exact* and is applicable to any affine transformation *regardless of the magnitude of the deformation*.

1. INTRODUCTION

This paper is concerned with the general problem of object recognition and registration based on a set of known templates. However, while the set of templates is known, the variability associated with the object, such as its location and pose in the observed scene, or its deformation are unknown *a-priori*, and only the group of actions causing this variablity in the observation, can be defined. This huge variability in the object signature (for any single object) due to the tremendous set of possible deformations that may relate the template and the observed signature, makes any detection and recognition problem ill-defined unless this variablity is taken into account. In other words, implicit or explicit registration of the observed object signature with respect to any template in an indexed set is an inherent and essential part of the solution to any detection and recognition problem.

To enable a rigorous treatment of the problem we begin by defining the "similarity criterion". Let G be a group and S be a set (a function space in our case), such that G acts as a transformation group on S. The action of G on S is defined by $G \times S \to S$ such that for every $\phi \in G$ and every $s \in S$, $(\phi, s) \to s \circ \phi$ (composition of functions on the right), where $s \circ \phi \in S$. From this point of view, given two functions h and g on the same orbit, the initial task (that enables recognition in a second stage), is to find the element ϕ in G that makes h and g identical in the sense that $h = g \circ \phi$.

In this paper we concentrate on parametric modeling and estimation of affine transformations, which is a special case of the general problem of modeling the homeomorphism group. Theoretically, in the absence of noise, the solution to the recognition problem is obtained by applying each of the deformations in the group to the template, followed by comparing the result to the observed realization. In the absence of noise, application of one of the deformations to the template yields an image, identical to the observation. Thus the procedure of searching for the deformation that transforms g into h is achieved, in principle, by a mapping from the group (the affine group, in our case) to the space of functions defined by the orbit of g. However, as the number of such possible deformations is infinite, this direct approach is computationally prohibitive. Hence, more sophisticated methods are essential.

2. ESTIMATION OF MULTIDIMENSIONAL AFFINE TRANSFORMATIONS: PROBLEM DEFINITION

The basic problem addressed in this paper is the following: Given two bounded, Lebesgue measurable functions h, g with compact supports, and with no affine symmetry, as rigorously defined below, such that $h : R^n \to R$, $g : R^n \to R$ where

$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{c}) \ , \quad \mathbf{A} \in GL_n(R),\ \mathbf{x}, \mathbf{c} \in R^n \qquad (1)$$

find the matrix $\mathbf{A}$ and the translation vector $\mathbf{c}$.

Let $M(R^n, R)$ denote the space of compact support, bounded, and Lebesgue measurable functions from R^n to R. Let $N \subset M(R^n, R)$ denote the set of measurable functions with an affine symmetry (or affine invariance), *i.e.*, $N =$

$\{f \in M(R^n, R) | \exists \mathbf{A} \in \mathrm{GL}_n(R), \mathbf{c} \in R, (\mathbf{A}, \mathbf{c}) \neq (\mathbf{I}, \mathbf{0})$ such that $f(\mathbf{x}) = f(\mathbf{Ax} + \mathbf{c})$ for every $\mathbf{x} \in R^n\}$. Let $M_{Aff}(R^n, R) \triangleq M(R^n, R) \setminus N$ denote the set of compact support and bounded Lebesgue measurable functions with no affine symmetry.

In the following sections we show that the problem of finding the parameters of the unknown affine transformation, whose direct solution requires a highly complex search in a function space, can be formulated as a *parameter estimation problem*. Moreover, it is shown that the original problem can be formulated in terms of an *equivalent* problem which is expressed in the form of a *linear* system of equations in the unknown parameters of the affine transformation. A solution of this linear system of equations provides *the* unknown transformation parameters.

3. AN ALGORITHMIC SOLUTION

In this section we provide a constructive proof showing that given an observation on $h(\mathbf{x}) \in M_{Aff}(R^n, R)$ and an observation on $g(\mathbf{x}) \in M_{Aff}(R^n, R)$ where $h(\mathbf{x}) = g(\mathbf{Ax} + \mathbf{c})$, $\mathbf{A}$ and $\mathbf{c}$ can be *uniquely* determined. Moreover, it is shown that almost always the solution for the unknown parameters of the affine transformation is obtained by solving only a *linear* system of equations.

Let, $\mathbf{x}, \mathbf{y} \in R^n$, *i.e.*, $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$. Thus,

$$\mathbf{y} = \mathbf{Ax} + \mathbf{c}\ , \ \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} + \mathbf{b} \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{A}^{-1} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

while $\mathbf{c}$ and $\mathbf{b} = -\mathbf{A}^{-1}\mathbf{c}$ are n-dimensional vectors of unknown constants, each representing the translation along a different axis, in the coordinate transformation model and its inverse, respectively. More specifically let $\mathbf{b} = [q_{10}, q_{20}, \ldots, q_{n0}]^T$. Define $y_0 = 1$ and let $\tilde{\mathbf{y}} = [y_0, y_1, \ldots, y_n]^T$. Hence, using (2)

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{y}} \tag{3}$$

where $\mathbf{T}$ is an $n \times (n+1)$ matrix given by

$$\mathbf{T} = \begin{bmatrix} q_{10} & q_{11} & \cdots & q_{1n} \\ \vdots & & \ddots & \vdots \\ q_{n0} & q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

Since $\mathbf{A} \in GL_n(R)$, also $\mathbf{A}^{-1} \in GL_n(R)$. It is therefore possible to solve for $\mathbf{A}^{-1}$ and the solution for $\mathbf{A}$ is guaranteed to be in $GL_n(R)$. Moreover, as shown below, in the proposed procedure the transformation determinant is evaluated first, and by a different procedure than the one employed to estimate the elements of $\mathbf{A}^{-1}$. Hence, a non-zero Jacobian guarantees the existence of an inverse to the transformation matrix.

Let $f \in M_{Aff}(R^n, R)$ and let μ_n denote the Lebesgue measure on R^n. Define the notation

$$\int_{R^n} f = \int_{R^n} f d\mu_n$$

Note that in the following derivation it is assumed that the functions are bounded and have compact support, as they are measurable but not necessarily continuous. It is further assumed that $\mathbf{A} \in GL_n(R)$ has a positive determinant.

The first step in the solution is to find the Jacobian of the linear transformation $\mathbf{A}$. Assume that $\mathbf{c} = 0$. A simple approach is to evaluate the Jacobian through the identity relation:

$$\int_{R^n} h^2(\mathbf{x}) = \int_{R^n} g^2(\mathbf{Ax}) = \left|\mathbf{A}^{-1}\right| \int_{R^n} g^2(\mathbf{y}) \tag{4}$$

or through similar identities. Hence,

$$\left|\mathbf{A}^{-1}\right| = \frac{\int_{R^n} h^2(\mathbf{x})}{\int_{R^n} g^2(\mathbf{y})} \tag{5}$$

and $\left|\mathbf{A}^{-1}\right| = |\mathbf{A}|^{-1}$. The Jacobian of the transformation (3) is the same as in the case where there is no translation.

In the second stage we prove that, provided that g is "rich" enough in a sense we rigorously define below, $\mathbf{T}$ can be *uniquely* estimated by establishing an $n+1$ dimensional system of linear equations. More specifically, let $(\mathbf{T})_k$ denote the kth row of $\mathbf{T}$. Applying a family of Lebesgue measurable, left-hand compositions $\{w_\ell\} : R \to R$ to the known relation $h(\mathbf{x}) = g(\mathbf{Ax} + \mathbf{c})$ and integrating over both sides of the equality, we obtain

$$\begin{aligned} \int_{R^n} x_k w_\ell \circ h(\mathbf{x}) &= \left|\mathbf{A}^{-1}\right| \int_{R^n} (\mathbf{T}_k \tilde{\mathbf{y}}) w_\ell \circ g(\tilde{\mathbf{y}}) \\ &= \left|\mathbf{A}^{-1}\right| \sum_{i=0}^{n} q_{ki} \int_{R^n} \tilde{y}_i w_\ell \circ g(\tilde{\mathbf{y}}) \end{aligned} \tag{6}$$

Let

$$\mathbf{G} = \begin{bmatrix} \int_{R^n} w_1 \circ g(\mathbf{y}) & \int_{R^n} y_1 w_1 \circ g(\mathbf{y}) & \cdots & \int_{R^n} y_n w_1 \circ g(\mathbf{y}) \\ \vdots & \ddots & & \vdots \\ \int_{R^n} w_{n+1} \circ g(\mathbf{y}) & \int_{R^n} y_1 w_{n+1} \circ g(\mathbf{y}) & \cdots & \int_{R^n} y_n w_{n+1} \circ g(\mathbf{y}) \end{bmatrix}$$

Rewriting (6) in a matrix form, since $g(\mathbf{y}) = g(\tilde{\mathbf{y}})$,

$$\mathbf{G} \begin{bmatrix} q_{k0} \\ q_{k1} \\ \vdots \\ q_{kn} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| \int_{R^n} x_k (w_1 \circ h(\mathbf{x})) \\ \vdots \\ |\mathbf{A}| \int_{R^n} x_k (w_{n+1} \circ h(\mathbf{x})) \end{bmatrix} \tag{7}$$

Similar system of equations is solved for each k to obtain all the elements of $\mathbf{T}$. Hence we have the following:

Theorem 1 *Let $\mathbf{A} \in GL_n(R)$. Assume $h, g \in M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = g(\mathbf{Ax} + \mathbf{c})$. Given measurements of h and g, then $\mathbf{A}$ and $\mathbf{c}$ can be uniquely determined if there exists a set of Lebesgue measurable functions $\{w_\ell\}_{\ell=1}^{n+1}$ such that the matrix $\mathbf{G}$ defined above, is full rank.*

Remark: Note that the denominator of (5), as well as the elements of the matrix $\mathbf{G}$ depend only on the template and its coordinate system and thus have to be evaluated only *once*. Therefore, the denominator of (5) together with the matrix $\mathbf{G}$ represent all the information in the template, required for finding the affine transformation parameters including the translation. Hence, the denominator of (5) together with the matrix $\mathbf{G}$ form a **sufficient representation** of the template (similarly to the notion of sufficient statistics), so that the template itself is not needed for solving the estimation problem once $\mathbf{G}$ and the denominator of (5) have been evaluated.

Remark: It should be noted that although we use the term "estimation" throughout this paper, the solution in (7) for the affine transformation parameters is *exact* and is not an estimate in the usual sense of the word.

Remark: The application of the set $\{w_\ell\}$ to $g(\mathbf{y})$ yielding $\mathbf{G}$ is in fact a mapping from the space of compact support, bounded and measurable functions to the space of $(n+1)\times(n+1)$ matrices. Recall that any $(n+1)\times(n+1)$ matrix can be considered as a vector in $R^{(n+1)^2}$. As the set of singular $(n+1)\times(n+1)$ matrices is of measure zero in $R^{(n+1)^2}$ (see *e.g.*, [2]) it can be shown that unless g itself is *linearly non-informative*, *i.e.*, the first order moments of $w_p \circ g$ vanish for any w_p, there will *always* (in the almost sure sense) exist a set $\{w_\ell\}_{\ell=1}^{n+1}$ generating a non-singular matrix $\mathbf{G}$, and hence a solution for the elements of $\mathbf{T}$. In other words, for any g, which is "rich" enough there will always exist a set $\{w_\ell\}_{\ell=1}^{n+1}$ generating a non-singular matrix.

Remark: Note that the solution for $\mathbf{T}$ employs only zero (the Jacobian) and first order constraints (obtained by multiplying $w_\ell \circ h$ by x_k) and avoids the use of higher order moments. However, imposing such a restriction (which is clearly convienient due to its simplicity) may result in cases where a system of the type (7) does not exist, see [2]. It is then obvious that higher order moments are needed to obtain a system similar to (7) (yet nonlinear) with enough equations to solve for all the unknowns.

4. FINDING THE AFFINE TRANSFORMATION SUBJECT TO A SPATIALLY CONSTANT ILLUMINATION CHANGE

In the analysis carried out so far it has been assumed that there is no illumination variation between the template and the observation, and hence the observed deformation is only due to the geometric distortion of the coordinate system caused by the affine transformation. In this section we generalize the proposed solution and address the more general deformation model where the model given by (1) is replaced by

$$h(\mathbf{x}) = ag(\mathbf{A}\mathbf{x}+\mathbf{c})\,, \quad \mathbf{A}\in GL_n(R),\ \mathbf{x},\mathbf{c}\in R^n,\ \ a\in R,\ a>0 \tag{8}$$

where a, $\mathbf{A}$ and $\mathbf{c}$ are unknown and need to be determined. As we prove in this section, the problem of finding the constant illumination gain amounts to replacing the step in which the Jacobian of the transformation is being determined in the case where there is no illumination change between the observation and the template, by a step in which both the illumination change and the Jacobian are jointly determined. More specifically, since $h(\mathbf{x}) = ag(\mathbf{A}\mathbf{x}+\mathbf{c})$, we have

$$\int_{R^n} h^2(\mathbf{x}) = a^2 \int_{R^n} |\mathbf{A}^{-1}| g^2(\mathbf{y}) = a^2 |\mathbf{A}^{-1}| \int_{R^n} g^2(\mathbf{y}) \tag{9}$$

Similarly,

$$\int_{R^n} h^4(\mathbf{x}) = a^4 |\mathbf{A}^{-1}| \int_{R^n} g^4(\mathbf{y}) \tag{10}$$

Hence,

$$|\mathbf{A}^{-1}| a^2 = \frac{\int_{R^n} h^2(\mathbf{x})}{\int_{R^n} g^2(\mathbf{y})} \tag{11}$$

and

$$|\mathbf{A}^{-1}| a^4 = \frac{\int_{R^n} h^4(\mathbf{x})}{\int_{R^n} g^4(\mathbf{y})} \tag{12}$$

Thus, both the Jacobian, $|\mathbf{A}^{-1}|$, and the illumination gain, a, can be evaluated using (11)-(12). In the second stage $n+1$ linear and independent constraints of the type (6) on the elements of $\mathbf{T}$ must be set. Applying the family of Lebesgue measurable, left-hand compositions $\{w_\ell\} : R \to R$ to the known relation $\frac{1}{a}h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}+\mathbf{c})$ and integrating over both sides of the equality, similarly to (6) we obtain

$$\begin{aligned} \int_{R^n} x_k w_\ell \circ \left(\frac{h(\mathbf{x})}{a}\right) &= \left|\mathbf{A}^{-1}\right| \int_{R^n} (\mathbf{T}_k \tilde{\mathbf{y}}) w_\ell \circ\ g(\tilde{\mathbf{y}}) \\ &= \left|\mathbf{A}^{-1}\right| \sum_{i=0}^{n} q_{ki} \int_{R^n} \tilde{y}_i w_\ell \circ g(\tilde{\mathbf{y}}) \end{aligned} \tag{13}$$

and in a matrix form, since $g(\mathbf{y}) = g(\tilde{\mathbf{y}})$,

$$\mathbf{G} \begin{bmatrix} q_{k0} \\ q_{k1} \\ \vdots \\ q_{kn} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| \int_{R^n} x_k (w_1 \circ \frac{h(\mathbf{x})}{a}) \\ \vdots \\ |\mathbf{A}| \int_{R^n} x_k (w_{n+1} \circ \frac{h(\mathbf{x})}{a}) \end{bmatrix} \tag{14}$$

Thus, once the illumination gain and the Jacobian have been evaluated, the R.H.S. of (14) is known, and the solution for the elements of the kth row of $\mathbf{T}$ is obtained. Hence we have,

Theorem 2 *Let $\mathbf{A} \in GL_n(R)$. Assume $h, g \in M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = ag(\mathbf{A}\mathbf{x}+\mathbf{c})$, and a is an unknown real gain coefficient. Given measurements of h and g, then $\mathbf{A}$, $\mathbf{c}$ and a can be uniquely determined if there exists a set of Lebesgue measurable functions $\{w_\ell\}_{\ell=1}^{n+1}$ such that the matrix $\mathbf{G}$ is full rank.*

Remark: Similarly to the previous cases, where illumination is assumed fixed, the denominators of (11) and (12), as well as the elements of the matrix $\mathbf{G}$ depend only on the template and its coordinate system and thus have to be evaluated only *once*. Therefore, the denominators of (11), (12) together with the matrix $\mathbf{G}$ represent all the information in the template, required for finding the affine transformation parameters including the translation, in the case where the illumination gain is unknown. Hence, the denominators of (11) and (12) together with the matrix $\mathbf{G}$ form a sufficient representation of the template.

5. NUMERICAL EXAMPLE

The example illustrates the operation of the proposed algorithm on a car image. The template image dimensions are 3100×1200. It is shown in the bottom image of Figure 1. The observed deformed image is shown in the upper image of the figure. This image is an affine transformed version of the template and is also observed with lower illumination, such that the illumination gain is $a = 0.58$. The error in estimating the gain is $\hat{a} - a = 0.726 \cdot 10^{-7}$. The image coordinate system is $[-1, 1] \times [-1, 1]$. The translation vector is $[-0.4, -0.5]$ and the translation estimation error vector is $[1.85 \cdot 10^{-3}, 3.92 \cdot 10^{-5}]$. The deforming transformation is given by

$$\mathbf{A} = \begin{bmatrix} 0.4854 & 0.3527 \\ -0.3527 & 0.4854 \end{bmatrix}$$

where the estimate obtained by the proposed procedure is

$$\hat{\mathbf{A}} = \begin{bmatrix} 0.4722 & 0.3626 \\ -0.353 & 0.4858 \end{bmatrix} .$$

Finally, the estimated deformation is applied to the original template in order to obtain an estimate of the deformed object (middle image in Figure 1) which can be compared with the deformed observation shown in the upper image.

6. CONCLUSIONS

We have considered the problem of finding the affine transformation relating a given observation on a planar object with some pre-chosen template of this object. The direct approach for estimating the transformation is to apply each of the deformations in the affine group to the template in a search for the deformed template that matches the observation. We propose a method that employs a set of non-linear operators to replace this high dimensional problem by an *equivalent linear problem*, expressed in terms of the unknown affine transformation parameters. The proposed solution is *unique and exact* and is applicable to any affine transformation *regardless of the magnitude of the deformation.*

7. REFERENCES

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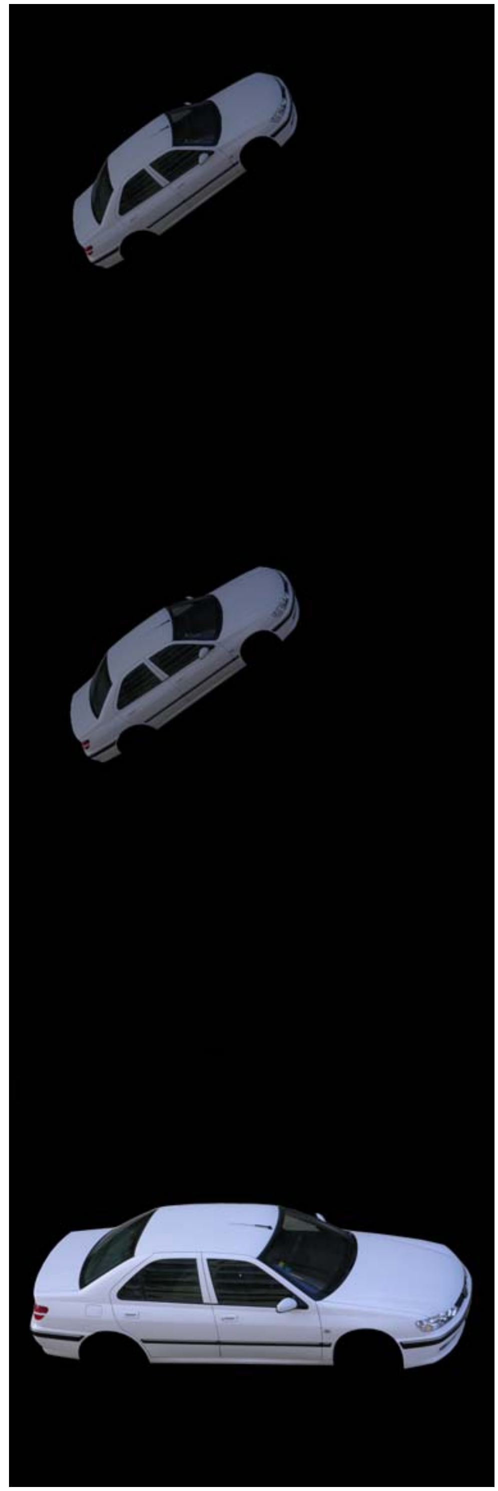

Figure 1: From bottom to top: Template; Estimated deformed object obtained by applying the deformation estimated from the observation to the template; Observation on the deformed object.