SNR ANALYSIS FOR PHASED-ARRAY MRI

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ABSTRACT

We develop principal components analysis for the optimal SNR phased-array magnetic resonance (MR) image recombination. As shown in our analysis, we can achieve the best possible SNR in both weak-noise and noisy cases, without needing to estimate the coil sensitivities or to remove noise effects using polynomial fitting or filtering. We provide both analysis as well as reconstruction techniques. Our results shed light on the performance of the phased-array image combination and give new insight into good image formation schemes.

1. INTRODUCTION

Phased-array magnetic resonance imaging (MRI) techniques have been developed for more than a decade, where a number of radio antennas, or coils, are used to receive and combine signals [1]. MR imaging systems with up to 16 receiver coils have been designed [2]. As a result, the imaging speed as well as image quality can be greatly improved, which benefits many MRI applications such as imaging and monitoring time-varying processes, and functional MRI. Because the coil sensitivities are difficult to model, especially for high field MRI, image reconstruction from multiple coils is a challenging problem.

The most commonly used method is the sum-of-square (SOS) reconstruction, or strictly speaking the root-mean-square (RMS) approach. This was introduced by Roemer *et al.* as a simple way of near-optimal combination without needing to know the coil sensitivities [3]. It has been noticed, however, the SOS introduces bias to the estimated images in the presence of noise, please see [4] and references therein. The bias is due to the rectification of noise during the RMS calculation and mainly affects the noise regions. Some research efforts have been directed to estimating the coil sensitivity signal and noise correlation matrices adaptively. Due to inherent difficulties in modeling the coil sensitivities, the estimation leads to bias or even lower composite image signal-to-noise ratio (SNR) than the SOS [5] [6] [7]. Recently, the performance in terms of SNR of some recombination techniques, such as the SOS, singular value decomposition and coil average have been analyzed [7]. The authors noticed that the average reconstruction method usually has a lower SNR than the other techniques. The SNR they considered, however, is to be shown still suboptimal. In this article, we perform principal components analysis (PCA) and construct an image reconstruction technique that achieves the best SNR. The motivation for this analysis is to model the phased-array MR imaging reconstruction in a way similar to the block-fading (piecewise constant) channels, which emulates slowly varying fading channels in multiple-antenna systems. Our paper takes a different approach from SENSE or TSENSE [8] [4], in that SENSE or TSENSE deals with coil sensitivity estimates and uses either polynomial fitting or filtering to achieve a high degree of alias artifact rejection in the presence of noise. In contrast, we do not handle coil sensitivity estimates. By exploiting the properties of coherence regions and constructing new matrices, we obtain image pixel values directly from eigen analysis, with the aim to achieve the highest SNR possible in the presence of noise, and to remove the pixelwise scaling errors. Our analyses shed light on the performance of the phased-array MRI reconstruction, and give new insight into good image formation schemes.

In this correspondence article, vectors are denoted by underlined small letters. We use capital letters to denote matrices (some constants are also capital letters, which should be clear from the context). The superscripts of $\star$, T, and H denote the complex conjugation, transpose, and Hermitian transpose, respectively. The notation $|\cdot|$ and $Re(\cdot)$ represent the magnitudes and real parts of complex numbers, respectively, and $\|\cdot\|$ represents the l_2 norms of vectors. $I_{L\times L}$ denotes the $L \times L$ identity matrix. $Tr(\cdot)$ denotes the trace of a matrix. The article will be organized as follows: Section 2 constructs the optimum SNR from Bayesian analysis. Section 3 provides the PCA analysis and construction. This article will be concluded in Section 4.

2. BAYESIAN ANALYSIS FOR OPTIMUM SNR

We will consider the MRI signal model defined in [10]. On a slice parallel to the (x, y) plane, define

$$k_x(t) = \int_0^t G_x(\tau)d\tau, \qquad k_y(t) = \int_0^t G_y(\tau)d\tau, \quad (1)$$

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where $G_x(\tau)$ and $G_y(\tau)$ are the strengths of the external magnetic gradient field along x and y axes. For a given antenna, the received signal without noise is

$$\tilde{s}(t) = e^{-\imath\omega_0 t} \int\int \rho(x,y)c^\star(x,y)e^{-\imath\lambda(k_x(t)x+k_y(t)y)}dxdy, \quad (2)$$

where $\rho(x,y)$ is the complex-valued MR contrast proportional to the transverse magnetization, which is regarded as the pixel value of the MR image, $c(x,y)$ is the coil sensitivity value. In the spatial domain, we denote the complex-valued received signal as

$$s(x,y) = \rho(x,y)c^\star(x,y). \quad (3)$$

For multiple arrays consisting of M coils, we consider an N-pixel image $\underline{\rho} = \{\rho_k\}_{k=1}^N$, where each element of $\underline{\rho}$ is an MR pixel. Let $C_N = [c_{k,l}]$ be the coil sensitivity matrix, where $c_{k,l}$ is the coil sensitivity for the kth pixel at the lth coil, $1 \le k \le N$, and $1 \le l \le M$. Let $S_N = [s_{k,l}]$ be the measurement matrix, which is described as [10]

$$s_{k,l} = \rho_k c_{k,l}^\star + \sigma n_{k,l}, \quad (4)$$

where $N_N = [n_{k,l}]$ is a complex-valued noise matrix, and σ is a scaling factor with σ^2 representing the noise power. The noise matrix satisfies $E(n_{k1,l1}^\star n_{k2,l2}) = q_{l1,l2}\delta_{k1,k2}$, where δ is the Kronecker delta function. That is, at a given pixel, the noise is correlated across coils; at different pixels, uncorrelated. The noise is modeled as Gaussian with zero mean and covariance matrix $Q = [q_{k,l}]$. Following the assumptions in [7], we also assume that the real and imaginary parts of the measurement noise are uncorrelated and with the same variance.

With the above model, when the coil sensitivities are known, the likelihood of the observations is

$$p(\underline{s}_k|Q,\underline{c}_k) = (2\pi)^{-\frac{N}{2}}|Q|^{-\frac{1}{2}}e^{-\frac{1}{2}(\underline{s}_k-\rho_k\underline{c}_k^\star)^H Q^{-1}(\underline{s}_k-\rho_k\underline{c}_k^\star)}, \quad (5)$$

where $\underline{s}_k^T$ and $\underline{c}_k^T$ are the kth row vectors of S_N and C_N, respectively. By taking the derivative of the log-likelihood and equating it to zero, we can obtain the maximum-likelihood estimation of the pixel value

$$\hat{\rho}_k = \frac{\underline{c}_k^T Q^{-1}\underline{s}_k}{\underline{c}_k^T Q^{-1}\underline{c}_k^\star}. \quad (6)$$

The derivation of this simple combination formula is straightforward. Even though optimal, it is not practical at all, because the coil sensitivities are usually unknown in practice. In this paper, we only use it as a theoretical reference in comparing SNR values of various schemes.

It can be easily shown that with known coil sensitivities, $\hat{\rho}_k$ is unbiased, that is, $E(\hat{\rho}_k) = \rho_k$. The variance of the estimator is $Var(\hat{\rho}_k) = \sigma^2/\,(\underline{c}_k^T Q^{-1}\underline{c}_k^\star)$. Thus, the SNR of this estimator is

$$SNR_o = \frac{|\rho_k|^2}{\sigma^2}\underline{c}_k^T Q^{-1}\underline{c}_k^\star. \quad (7)$$

Notice that for comparison purposes, we shall follow the definition of SNR used in [7], which is the squared signal level divided by the variance of noise. That is, this definition is the square of the ordinarily used SNR.

In contrast, in the same situation, the best SNR obtained in [7] is

$$SNR' = \frac{|\rho_k|^2\|\underline{c}_k\|^4}{\sigma^2\underline{c}_k^H Q\underline{c}_k}. \quad (8)$$

If Q is a uniform matrix, $Q = q \cdot I_{M\times M}$, q is a positive constant, it can be seen that $SNR_o = SNR'$. In the general case, however, it can be shown using the Cauchy-Schwartz inequality that $\underline{c}_k^T Q^{-1}\underline{c}_k^\star \ge \|\underline{c}_k\|^4/(\underline{c}_k^H Q\underline{c}_k)$, i.e., $SNR_o \ge SNR'$. To see this, decompose Q into $Q = U^T\Lambda U$, where $U^T U = I_{M\times M}$, and $\Lambda = Diag(\lambda_1,\cdots,\lambda_M)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_M > 0$. Now we have

$$\begin{aligned}(\underline{c}_k^T Q^{-1}\underline{c}_k^\star)(\underline{c}_k^H Q\underline{c}_k) &= (\underline{d}_k^T\Lambda^{-1}\underline{d}_k^\star)(\underline{d}_k^H\Lambda\underline{d}_k)\\ &= \sum_{l=1}^M \frac{|d_l|^2}{\lambda_l}\sum_{l=1}^M |d_l|^2\lambda_l \\ &\ge (\sum_{l=1}^M |d_l|^2)^2 = \|\underline{c}_k\|^4,\end{aligned} \quad (9)$$

where $\underline{d}_k = U\underline{c}_k$. The equality can be achieved only if $\lambda_1 = \cdots = \lambda_M = q$.

Making use of Kantorovich inequality [11], the following upper bound can be yielded:

$$\frac{SNR_o}{SNR'} = \frac{(\underline{c}_k^T Q^{-1}\underline{c}_k^\star)(\underline{c}_k^H Q\underline{c}_k)}{\|\underline{c}_k\|^4} \le \frac{(\lambda_1+\lambda_M)^2}{4\lambda_1\lambda_M}. \quad (10)$$

The above upper bound actually is a good approximation to the true value of SNR_o/SNR'. If the largest eigenvalue is thrice as large as the smallest eigenvalue, for example, the ratio of the SNRs, SNR_o/SNR', is approximately $\frac{4}{3}$.

According to the analysis of [7], the proposed methods, including the SOS, singular vector decomposition, and coil-average reconstruction achieve a performance with at most SNR'. These methods, therefore, are generally suboptimal compared to the optimum SNR_o. The coil sensitivities, however, are usually unknown a priori and hard to model in practice. Thus, the above maximum-likelihood estimator cannot be directly applied. Thanks to the property of the magnetic fields, the sensitivities are regarded as the fading coefficients of a slowly-varying channel. To adapt to individual coil field maps, we shall make use of the principal components analysis and piecewise-constant fading channel properties to combine the MRI images.

3. PCA OF PHASED-ARRAY MRI RECONSTRUCTION

As mentioned above, the coil sensitivities are hard to model in practice. The noise variance, on the contrary, can be estimated easily [6]. Thus, we shall assume that Q is known to us through adaptive estimation but the coil sensitivities C are unknown. In this case, we perform the PCA to estimate the MRI combined pixel values.

Before introducing the PCA method, let us consider the modified SOS method which uses $\sqrt{\underline{s}_k^H Q^{-1} \underline{s}_k}$ as the estimate of the MRI contrast value. It may be regarded as a weighted version of the SOS using Q matrix, which can be estimated in an adaptive fashion [6]. This estimate is a function of σ, denoted as $\hat{\rho}_k(\sigma)$. In the noise-free case, $\hat{\rho_k}(0) = |\rho_k|\sqrt{\underline{c}_k^T Q^{-1} \underline{c}_k^\star}$. In the noisy case, by using the first-order Taylor expansion, we can find the SNR to be:

$$SNR = \frac{(\hat{\rho_k}(0))^2}{\sigma^2 E(|\frac{d}{d\sigma}\hat{\rho}_k(0)|^2)} = \frac{|\rho_k|^2}{\sigma^2} \underline{c}_k^T Q^{-1} \underline{c}_k^\star. \qquad (11)$$

It can be seen that in small-noise scenarios, the SNR of the modified SOS achieves the optimality. The estimate, however, still comes with the scaling factor $\sqrt{\underline{c}_k^T Q^{-1} \underline{c}_k^\star}$. If the scaling factor is not constant, the estimate will have scaling errors across pixels. Even if $\underline{c}_k$ is a constant vector across different pixels, in general, the error still plagues the reconstruction due to the existence of inverse noise variance.

To alleviate from the scaling errors caused by $\sqrt{\underline{c}_k^T Q^{-1} \underline{c}_k^\star}$, let us consider the matrix constructed from the kth observation, $k = 1, \cdots, N$,

$$R := \underline{s}_k^\star \underline{s}_k^T = |\rho_k|^2 \underline{c}_k \underline{c}_k^H + 2\sigma Re(\rho_k^\star \underline{c}_k \underline{n}_k^H) + \sigma^2 \underline{n}_k^\star n_k^T. \qquad (12)$$

The noise level can be estimated through a standard prescan calibration. For weak noise, $R \approx |\rho_k|^2 \underline{c}_k \underline{c}_k^H$, and the eigenvector of R is $\underline{c}_k$ (up to a scaling factor), corresponding to the largest principal component $|\rho_k|^2 \|\underline{c}_k\|^2$. We can examine the principal eigenvector of R to determine if the coil sensitivity vector is approximately constant for different pixels. Because of the piecewise constant channel property, this condition usually holds for a coherence region.

Now let us consider a coherence region where $\underline{c}_k$ is approximately a constant vector to make use of the PCA. Assume for pixels $\underline{\rho} = [\rho_1, \cdots, \rho_L]^T$, this condition holds. Then we consider the model in Eq. (4) for L pixels in this small region to exploit the spatial diversity that the antenna array provides:

$$S_L = \underline{\rho}\underline{c}^H + \sigma N_L, \qquad (13)$$

where $\underline{c}$ is the coil sensitivity which is the same across L pixels, and N_L is an $L \times M$ noise matrix. After obtaining S_L, the whole picture S_N can be obtained by patching different S_Ls. We construct the following matrix

$$\begin{aligned} P &= S_L Q^{-1} S_L^H \\ &= (\underline{\rho}\underline{c}^H) Q^{-1} (\underline{c}\underline{\rho}^H) + 2\sigma Re(\underline{\rho}\underline{c}^H Q^{-1} N_L^H) + \sigma^2 N_L Q^{-1} N_L^H. \end{aligned} \qquad (14)$$

In the weak-noise case, P is approximately equal to the first term, $(\underline{\rho}\underline{c}^H)Q^{-1}(\underline{c}\underline{\rho}^H)$, whose principal eigenvector is $\underline{\rho}$ (up to a scaling factor), corresponding to the principal eigenvalue of $\|\underline{\rho}\|^2(\underline{c}^H Q^{-1} \underline{c})$. This weak-noise case can be related to the SVD method [7], where singular values are exploited to achieve an SNR that is the same as SOS. In [9], the authors have shown SOS can asymptotically achieve optimal SNR if $Q = I$. In general, SOS will lead to biased images as demonstrated in Section 2.

Analogously to multiple-antenna communications systems, if we make use of a training scheme so that a training sequence is employed to estimate $\underline{c}$ in this coherence region, then we can recover the power of $\|\underline{\rho}\|^2$, thus, reconstruct $\underline{\rho}$ without any scaling ambiguity. This training sequence can be designed optimally in a way similar to [12].

In the noisy case, taking the expectation of $S_L Q^{-1} S_L^H$ leads to

$$E(P) = (\underline{\rho}\underline{c}^H) Q^{-1} (\underline{c}\underline{\rho}^H) + M\sigma^2 I_{L\times L}. \qquad (15)$$

In the above, we have used

$$E(N_L Q^{-1} N_L^H) = M I_{L\times L}. \qquad (16)$$

To see this, we provide its derivation in the following. Denote N_L by $[\underline{n}_1, \cdots, \underline{n}_L]^T$, where $\underline{n}_k$ is an $M \times 1$ column vector, $k = 1, \cdots, L$. Let us consider $E(\underline{n}_i^T Q^{-1} \underline{n}_j^\star)$. In the case $i \neq j$, according to our assumption on the noise, $E(\underline{n}_i^T Q^{-1} \underline{n}_j^\star) = 0$. In the case $i = j$, we have

$$\begin{aligned} E(\underline{n}_i^T Q^{-1} \underline{n}_i^\star) &= E(Tr(Q^{-1} \underline{n}_i^\star \underline{n}_i^T)) \\ &= Tr(E(Q^{-1} \underline{n}_i^\star \underline{n}_i^T)) \\ &= Tr(I_{M\times M}) = M. \end{aligned}$$

In the above, we have used the matrix identity. Thus, $E(N_L Q^{-1} N_L^H) = E([\underline{n}_i^T Q^{-1} \underline{n}_j^\star]_{L\times L}) = M I_{L\times L}$.

Now, we can observe the principal components of $E(P)$ consisting of one large eigenvalue, $\lambda_{0,P} := \|\underline{\rho}\|^2(\underline{c}^H Q^{-1} \underline{c}) + M\sigma^2$, and $L-1$ smaller eigenvalues $\lambda_{1,P} := M\sigma^2$. The eigenvectors of $E(P)$ are exactly the same as those of $(\underline{\rho}\underline{c}^H)Q^{-1}(\underline{c}\underline{\rho}^H)$. The dominant eigenvector corresponding to $\lambda_{0,P}$ is $\underline{\rho}$ (up to a scaling factor), whose elements are exactly the pixel values of the MRI in this region. It can be seen $\underline{\rho}$ may serve as a maximum eigenfilter that is the stochastic counterpart of a matched filter and maximizes the output SNR for a random signal. Denote the normalized eigenvector $\underline{\rho}$ as $\underline{\rho}_0$, that is, $\|\underline{\rho}_0\| = 1$. In order to get the true value of $\underline{\rho}$, we need additional information regarding the power of $\underline{\rho}$. Again, training sequences may be employed for this purpose. The noise power σ^2 can be obtained by simply dividing the smaller eigenvalue by the number of coils. Without any additional information, we use the following estimation:

$$\rho_0 \sqrt{\lambda_{0,P} - \lambda_{1,P}}. \tag{17}$$

In the case of multiple scans, with the assumption of ergodicity, the expectation in (15) can be evaluated using the consecutive scans of the object. Since typically averaging of SOS reconstructed images over a large number of scans is done is practice, evaluating the expectation is as practical as the SOS procedure.

The estimation and perturbation of $\lambda_{0,P}$ lead to the SNR of $\frac{\|\underline{\rho}\|^2}{L\sigma^2}\underline{c}^H Q^{-1}\underline{c}$. Since $\|\underline{\rho}\|^2 = \sum_{k=1}^{L} |\rho_k|^2$, the SNR becomes the average of the pixelwise optimum SNRs in the region considered. Most importantly, the pixels reconstructed in the coherence region are free from the nonuniform scaling errors.

4. CONCLUSION

This article considers the principal components analysis of phased-array MR image recombination. The pixelwise optimum SNR is obtained. In both weak-noise and noisy scenarios, the method obtains the optimum SNR, at the same time, alleviates from the nonuniform scaling errors. Our future research line includes conducting experiments using real objects and simulations to demonstrate the power of the optimum reconstruction technique. This analysis sheds light on the performance of phased-array MRI reconstruction, and gives new insight into good MR image formation schemes.

5. REFERENCES

[1] J.W. Carlson and T. Minemura, "Imaging time reduction through multiple receiver coil data acquisition and image reconstruction," *Magnetic Resonance in Medicine*, vol. 29, pp. 681-688, 1993.

[2] J.R. Porter, S.M. Wright, and A. Reykowski, "A 16-element phased-array head coil," *Magnetic Resonance in Medicine*, vol. 40, pp. 272-279, 1998.

[3] P.B. Roemer, W.A. Edelstein, C.E. Hayes, S.P. Souza, and O.M. Mueller, "The NMR phased array," *Magnetic Resonance in Medicine*, vol. 16, no. 2, pp. 192-225, 1990.

[4] P. Kellman, F.H. Epstein, and E.R. McVeigh, "Adaptive sensitivity encoding incorporating temporal filtering (TSENSE)," *Magnetic Resonance in Medicine*, vol. 45, no. 5, pp. 846-852, 2001.

[5] M. Bydder, D.J. Larkman, and J.V. Hajnal, "Combination of signals from array coils using image-based estimation of coil sensitivity profiles," *Magnetic Resonance in Medicine*, vol. 47, pp. 539-548, 2002.

[6] D.O. Walsh, A.F. Gmitro, and M.W. Marcellin, "Adaptive reconstruction of phased array MR imagery," *Magnetic Resonance in Medicine*, vol. 43, pp. 682-690, 2000.

[7] D. Erdogmus, E.G. Larsson, R. Yan, J.C. Pricipe, and J.R. Fitzsimmons, "Asymptotic SNR-performance of some image combination techniques for phased-array MRI," *Signal Processing*, vol. 84, pp. 997-1003, July 2004.

[8] K.P. Pruessmann, M. Weiger, M.B. Scheidegger, and P. Boesiger, "SENSE: sensitivity encoding for fast MRI," *Magnetic Resonance in Medicine*, vol. 42, no. 5, pp. 952-962, 1999.

[9] E.G. Larsson, D. Erdogmus, R. Yan, J.C. Pricipe, and J.R. Fitzsimmons, "SNR-optimality of sum-of-squares reconstruction for phased-array magnetic resonance imaging," *Journal of Magnetic Resonance*, vol. 163, no. 1, pp. 121-123, 2003.

[10] E.M. Haacke, R.W. Brown, M.R. Thompson, R. Venkatesan, *Magnetic Resonance Imaging-Physical Principles and Sequence Design*, New York: Wiley, 1999.

[11] D.G. Luenberger, *Optimization by Vector Space Methods*. New York: Wiley, 1969.

[12] B. Hassibi and B.M. Hochwald, "How much training is needed in multiple-antenna wireless links?" *IEEE Trans. Inform. Theory*, vol. 49, pp. 951-646, Apr. 2003.